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A FEEDBACK CONTROL MODEL FOR NETWORK FLOW WITH MULTIPLE PURE TIME DELAYS

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ABSTRACT

A control model describing a network flow hindered by multiple pure time (or transport) delays is formulated. Feedbacks connect each desired output with a single control sector situated at the origin. The dynamic formulation invokes the use of differential-difference equations. This causes the characteristic equation of the model to consist of transcendental functions instead of a common algebraic polynomial. A general graphical criterion is developed to evaluate the stability of such a problem since the literature is evasive in the case of multiple coupled delays. A digital computer simulation later confirms the validity of such criterion. An optimal decision-making process with multiple delays serves as an application of the analytical effort.

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SECTION I

INTRODUCTION

A. PROCESSES WITH MEMORY AND ANTICIPATION; THE DIFFERENTIAL-DIFFERENCE EQUATION

In concept, there exists a wide class of mathematical processes which operate in the present using knowledge acquired from the past as well as information obtained from the future. This reveals the presence of memory, the ability to remember the past, as well as anticipation, the ability to predict the future. If these processes are to be realizable in our world then they must obey the law of causation: the cause of things must come before their effect. A chronological order is imposed and the processes, in this case, "follow their own destiny." The capacity to anticipate is suppressed and only present and past states are now considered. Despite this restriction, the realization of a man-made process or system* operating in the present using knowledge from the past is an elegant concept since it simulates the way humans learn and think.

Processes with memory can be expressed mathematically by an interesting class of equations: the differential-difference equations, involving derivatives along with differences. The literature on such special mathematics embraces many disciplines and spreads over three centuries in research. The earliest work available is that of Bernoulli,^{1†} who in 1728 was studying oscillatory motion of strings. In applied mathematics, famous names like Lagrange², Poisson³ and Cauchy⁴ are often associated with equations of differential-difference form. They are reflected by today's contemporaries in this field such as Bellman and Cooke⁵, Pinney⁶, Ogutzoreli⁷, and Wright⁸. Phenomena possessing memory have also been studied in the physical sciences by Bateman⁹ in radioactivity, Gerasimov¹⁰ in heat conduction, Lagrange¹¹ in sound propagation, Arley¹² in cosmic radiations, Minorsky¹³, Nisolle¹⁴, Volterra¹⁵, Picard¹⁶, and Satche¹⁷ all in mechanics and elasticity. In biology, one finds differential-difference equations applied to the renewal process¹⁸⁻¹⁹, natural and artificial selection²⁰, and the fight for survival²¹.

*For convention purposes, appendices A and B provide basic definitions for the word system and its associated prefixes, such as feedback and control, in context with this work. Figure 1 illustrates these definitions.

†A superscript indicates a reference number appearing in the bibliography.

Economists have also made contributions with these special equations. To mention a few, Frish²² in economic dynamics, Kalecki²³ in business cycles, James and Belz,²⁴ Samuelson²⁵, Bateman²⁶ in retail trade theory, Theiss²⁷ in the study of savings and investments, and Tinbergen²⁸ in the modeling of ship-building cycles are all notable names.

B. THE DELAY PROBLEM

In modern times, operational models with transport or time delays form the new vocabulary for procedures governed by differential-difference equations. A delay means that one or more operations are temporarily suspended in time due to some constraints.

In engineering, one can cite numerous references on time-delay problems: Tsien²⁹ in rocketry, also Smith³⁰ in electrical systems, Truxal³¹ in electrical signals, Rogers and Connolly³² in analog computers, Huggins³³ in systems dynamics, Oetker³⁴, Paul³⁵, Caughanowe and Koppel³⁶, and Tyner³⁷ all in process control for the chemical industry. Most recently, the time-delay problem has found application in the theory of artificial intelligence.³⁸

C. THE DELAY PROBLEM AND THE MANAGEMENT FUNCTION

As far as management analysis is concerned, it is well-known that practical trends in the U.S. are brought about by two schools of thought: those who practice the case method, relying on past cases, and those who favor the quantitative approach, relying on analytical knowledge gained by studying physical systems. The first method is reflected by the Harvard circles while the second one is practiced by the Stanford and MIT groups. Forrester³⁹ in applying the latter method at MIT has stressed the importance of time delays in industrial dynamics such as production, inventory and sales.

The new field of Operations Research, still part of the quantitative approach, renders time-delay problems imminent firstly in the network flow analysis where minimum transition time is usually the objective and secondly in Queueing and Inventory Theory⁴⁰. Prabhu⁴¹ has considered similar mathematics in his work on models for dams and grain storage.

D. THE GENERAL OBJECTIVE OF THIS RESEARCH

In the real world, autonomous organizations are confronted with the daily task of applying continuous improvement mechanisms in order to enhance the

viability of their working processes. Such strategy involves tactical efforts on the part of decision-makers. Faced with commitments and schedules, the management firstly evaluates the input resources at hand, secondly applies a planned intervention, and finally seeks feed-back, which is often delayed in time, in order to determine the effectiveness of the motion taken.

Such a philosophical management model has possible application in the following:

1. Transportation networks suffering delays in schedule.
2. The administration of federal, state or local programs where a complicated bureaucracy often delays needed resources such as monetary funds.
3. Administrative control of the economy such as the imposition of ceilings on prices and wages. In this case, feedback of the results created is heavily delayed due to the complexity of the economic sector.

This paper intends to highlight the above applications through mathematical modeling. The effort illustrates the decision-making process involved in a network flow containing information and resource channels hindered by coupled pure time delays.

SECTION II

MATHEMATICAL BACKGROUND

A. ASSUMPTIONS LEADING TO THE DIFFERENTIAL-DIFFERENCE EQUATION

One can interpret processes with memory and anticipation using the following general functional equation:

$$\begin{aligned} \Phi [\vec{y}(\vec{x}, t)] &= f \{ \vec{y}(t), \vec{y}(t \pm t_1(t)), \\ &\vec{y}(t \pm t_2(t)), \dots, \vec{y}(t \pm t_n(t)), \\ &\vec{x}(t), \vec{x}(t \pm t_1(t)), \vec{x}(t \pm t_2(t)), \dots \\ &\vec{x}(t \pm t_n(t)) \}, \end{aligned} \quad (1)$$

where Φ is a generalized n^{th} order ($n > 0$) linear or non-linear, differential or difference, partial or ordinary operator. \vec{y} and \vec{x} are vectors of the various dependent and independent variables (functions of time) respectively. t_1, t_2, \dots, t_n represent backward or forward delays in time, themselves possible functions of time. Terms like $\vec{y}(t \pm t_n(t))$ signify that \vec{y} is evaluated at a time t shifted in the future by $+t_n(t)$, or in the past by $-t_n(t)$. Such a formulation must also be accompanied with proper initial, boundary and constraint conditions.

Unfortunately, no tools are available to fully treat eq. (1). Simplifying assumptions must be made. If one is restricted to a realizable, i.e. non-anticipatory, linear, i.e. the principle of superposition holds, deterministic (non-probabilistic) system and furthermore, if the dependent variables are functions of time only, and having constant coefficients along with constant delays then eq. (1) reduces to the following differential-difference equation:

$$D[\vec{x}(t)] = f \{ \vec{x}(t - t_1), \vec{x}(t - t_2), \dots, \vec{x}(t - t_n), \vec{x}(t), t \} \quad (2)$$

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where

$$D [] = \left[a_n \frac{d^n}{dt^n} + a_{n-1} \frac{d^{n-1}}{dt^{n-1}} + \cdots + a_1 \frac{d}{dt} \right]$$

is the linear ordinary differential operator, with the a's being constant coefficients.

If, in addition, the process is instantaneously dependent on just the present state, then eq. (2) further reduces to the ordinary differential equation which traditionally governs countless mathematical models:

$$D [\vec{x} (t)] = f (\vec{x}, t). \quad (3)$$

Eq. (3) is indeed restricted when compared to eq. (1).

Equation (2), which assumes present and past states, is expressed explicitly for a single input-output system as:

$$\begin{aligned} a_n \frac{d^n}{dt^n} x(t) + a_{n-1} \frac{d^{n-1}}{dt^{n-1}} x(t) + \cdots + a_1 \frac{d}{dt} x(t) = b_0 x(t) + b_1 x(t - t_1) + b_2 x(t - t_2) \\ + \cdots + b_n x(t - t_n) + u(t), \end{aligned} \quad (4)$$

where $u(t)$ is a prescribed input activating the process. The a's and b's are constant coefficients.

B. THE LAPLACE TRANSFORM OF A DIFFERENTIAL-DIFFERENCE EQUATION

Since Eq. (4) is linear it can be transformed to a simple polynomial form using the well-known Laplace Transform, \mathcal{L} . Remembering that

$$\mathcal{L} \left[\frac{df(t)}{dt} \right] = s \quad \text{and} \quad \mathcal{L} [f(t - t_1)] = e^{-st_1},$$

Eq. (4) becomes:

$$a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s = b_0 + b_1 e^{-st_1} + b_2 e^{-st_2} + \dots b_n e^{-st_n} + \mathcal{L}\{u(t)\} \quad (5)$$

(e = 2.7...)

One obtains the characteristic equation of the system that eq. (4) governs by rearranging eq. (5) and suppressing the input. Therefore,

$$F(s, e^s) = a_n s^n + a_{n-1} s^{n-1} + \dots a_1 s - b_0 - b_1 e^{-st_1} - b_2 e^{-st_2} - \dots b_n e^{-st_n} = 0 \quad (6)$$

is called characteristic to the system because it describes the free behavior, the outside input being suppressed. This equation helps determine stability, a salient feature in any system study, as later shown.

C. THE MATHEMATICAL FORM OF THE PURE TIME DELAY

Terms containing e^{-s} in eq. (6) correspond to the various transport lags or delay terms such as $y(t - t_i)$ present in eq. (4). The syntax of the word "time-delay" is often misleading due to the existence of numerous synonyms. Depending on the field of study, the following definitions occur in the literature:

- (1) distance-velocity lag (Mechanics)
- (2) retarded action (Mechanics)
- (3) dead time (in Electrical Engineering)
- (4) transport lag or delay (in Transportation Science)
- (5) pipe-line delay (in Chemical Engineering)
- (6) iddle period (Operations Research)

- (7) pure time delay (Applied Mathematics)
- (8) retarded argument (Applied Mathematics)
- (9) translation operator (Applied Mathematics)

They all refer to the same transformation:

$$y(t) \rightarrow \left[\begin{array}{c} \text{pure-time delay} \\ \text{of period } t_1 \end{array} \right] \rightarrow y(t - t_1)$$

The confusion arises when another function is mentioned: the phase or time lag. The latter is called, at a loss, an exponential delay.³⁹ Various such transformations are illustrated in Figure 2. The first-order time lag $1/(s\tau + 1)$, for instance, produces an output at the instant the input activates it. The pure time delay e^{-st_1} , on the other hand, does not respond for a time period t_1 after the input activates it, and only then, the response comes out as the exact image of the input.

This work will adopt the following notation:

$$\mathcal{L}[f(t - t_1)] = e^{-st_1} = \text{pure time delay (which concerns most of this research)}$$

$$\mathcal{L}[e^{-t/\tau_1}] = \frac{1}{s\tau_1 + 1} = \text{simple time lag (1st order)}$$

SECTION III

THE NETWORK FLOW GRAPH METHOD FOR PROCESSES WITH PURE TIME DELAYS

A. FROM THE DIFFERENTIAL-DIFFERENCE EQUATION TO A FLOW GRAPH

Instead of working directly with a differential-difference equation it is possible to manipulate its Laplace equivalent, in a graphical form. As an illustration, consider the following coupled system of differential-difference equations having a pure time delay and a simple time lag:

$$\dot{x}_1(t) = a \text{ prescribed function, } f(t), \quad (7)$$

serving as an input.

$$\dot{x}_2(t) = \dot{x}_1(t) - a x_6(t) \quad (8)$$

$$\dot{x}_3(t) = K \dot{x}_2(t) \quad (9)$$

$$\dot{x}_4(t) = e^{-t/\tau} \dot{x}_3(t) \quad (10)$$

(time lag)

$$x_5(t) = x_4(t) = \int \dot{x}_4(t) dt \quad (11)$$

$$x_6(t) = \underset{\text{(pure time delay)}}{x_5(t - t_1)} \quad (12)$$

($\dot{}$) signifies d/dt. K, A, τ and t_1 are constants. The Laplace transforms to equations (7-12) are:

$$X_1(s) = F(s) \quad (13)$$

$$s X_2 (s) = s X_1 (s) - a X_6 (s) \quad (14)$$

$$s X_3 (s) = K s X_2 (s) \quad (15)$$

$$s X_4 (s) = \frac{1}{\tau s + 1} \cdot s X_3 (s) \quad (16)$$

$$X_5 (s) = X_4 (s) = \frac{1}{s} \cdot s X_4 (s) \quad (17)$$

$$X_6 (s) = e^{-st_1} X_5 (s) \quad (18)$$

Utilizing the flow graph method exposed in Appendix C, eqs. (13-18) can be represented by the network flow appearing in Figure 3a. Such a graph consists of three major terms, shown in a brief version in Figure 3b as:

$$H (s) = H = \frac{K}{\tau s + 1} \cdot \frac{1}{s} = \left\{ \begin{array}{l} \text{the series product of all the} \\ \text{terms in cascade excluding} \\ \text{the pure time delay,} \end{array} \right\} \quad (19)$$

$$D (s) = D = e^{-st_1} = \text{the pure time delay,} \quad (20)$$

$$G (s) = G = - a = \text{the feedback loop.} \quad (21)$$

B. THE TRANSFER FUNCTION FOR A SINGLE PURE TIME DELAY PROBLEM

Applying Mason's Loop Rule (see Appendix C) it is possible to obtain the transfer function (T.F.) defined in Appendix B for the system in Figure 3b as follows:

$$P_1 = H D \quad (22)$$

$$L_1 = - H D G \quad (23)$$

$$\Delta = 1 - L_1 = 1 + H D G \quad (24)$$

$$\Delta_1 = 1 \quad (25)$$

where P_1 is the only forward path, L_1 is the only feedback loop factor, Δ is the determinant, and Δ_1 is the cofactor. Thus

$$T. F. = \sum_i^{N=1} \frac{P_i \Delta_i}{\Delta} = \frac{H D}{1 + H D G} \quad (26)$$

Setting the denominator of equation (26) equal to zero, yields the characteristic equation for the process:

$$\begin{aligned} 1 + H D G &= 1 + H(s) \cdot D(s) \cdot G(s) \\ &= 1 - \frac{a K}{(\tau s + 1)} \cdot \frac{1}{s} \cdot e^{-st_1} = 0 \end{aligned} \quad (27)$$

(C) THE CASE OF MULTIPLE TIME DELAYS:

Consider a more complicated case. One can allow for two pure time delays as shown in Figure 4a. For this case one has, again for Mason's Loop Rule:

$$P_1 = H_1 D_1 \quad (28)$$

$$P_2 = H_2 D_2 \quad (29)$$

$$L_1 = - H_1 D_1 G_1 \quad (30)$$

$$L_2 = - H_2 D_2 G_2 \quad (31)$$

$$\Delta = 1 - (L_1 + L_2) = 1 + H_1 D_1 G_1 + H_2 D_2 G_2 \quad (32)$$

$$\Delta_1 = 1 \quad (33)$$

$$\Delta_2 = 1 \quad (34)$$

and the T.F. is:

$$T. F. = \frac{H_1 H_2 D_1 D_2}{1 + H_1 D_1 G_1 + H_2 D_2 G_2} \quad (35)$$

This problem can be said to be a trivial extension of the single delay case. This is so because it can be separated into two single delay problems as shown in Figure 4b. The two loops are not coupled since the output of the first system is directly used as the input to the second one and the latter has no feedback influence on the first one. If coupling is present however then the system is not trivial. Figure 4c shows the non-trivial case for two delays. For this flow graph, one has:

$$P_1 = H_1 H_2 D_1 D_2 \quad (36)$$

$$P_2 = 0 \quad (37)$$

$$L_1 = - H_1 D_1 G_1 \quad (38)$$

$$L_2 = - H_1 H_2 D_1 D_2 G_2 \quad (39)$$

$$\Delta = 1 - (L_1 + L_2) = 1 + H_1 D_1 (G_1 + H_2 G_2 D_2) \quad (40)$$

$$\Delta_1 = 1 \quad (41)$$

and therefore,

$$T. F. = \frac{\sum_{i=1}^2 P_i \Delta_i}{\Delta} = \frac{H_1 H_2 D_1 D_2}{1 + H_1 D_1 (H_2 G_2 D_2 + G_1)} \quad (42)$$

Extending the work to a non-trivial problem with N delays, the flow is shown in Figure 4d. Its general T.F. can be shown to take the form:

$$T. F. = \frac{\prod_{i=1}^N H_i D_i}{1 + \sum_{j=1}^N \left[G_j \prod_{i=1}^j H_i D_i \right]} \quad (43)$$

where

$\sum_{j=1}^N$ = summation of terms over the j index from 1 to N,

$\prod_{i=1}^N$ = product of terms over the i index from 1 to N.

$D_i = e^{-st_i}$ (pure time delay)

G_j = feedback terms

H_i = product of all terms in each i^{th} forward path between nodes, excluding each D_i .

If one sets the denominator of equation (43) equal to zero then the result yields the characteristic equation for the general non-trivial case:

$$F(s, e^s) = 1 + \sum_{j=1}^N \left[G_j \prod_{i=1}^j H_i D_i \right] = 0 \quad (43a)$$

with G_j and F_i containing algebraic terms in s, and D_i containing the exponential terms e^{-st_i} . As an illustration, if three non-trivial pure time delays are present then equation (43a) takes the form for $N = 3$:

$$\begin{aligned} F(s, e^s) &= 1 + G_1 H_1 D_1 + G_2 H_1 H_2 D_1 D_2 + G_3 H_1 H_2 H_3 D_1 D_2 D_3 \\ &= 1 + G_1(s) H_1(s) \cdot e^{-st_1} + G_2(s) H_1(s) H_2(s) \cdot e^{-s(t_1+t_2)} \\ &\quad + G_3(s) H_1(s) H_2(s) H_3(s) \cdot e^{-s(t_1+t_2+t_3)} = 0 \end{aligned} \quad (43b)$$

SECTION IV

THE STABILITY OF PROCESSES

HAVING PURE TIME DELAYS

A. DEFINITIONS

A linear system is defined as stable if the time function representing its response to a simple impulse input remains bounded in amplitude as $t \rightarrow \infty$, and unstable otherwise. More important, a stable system is asymptotically stable if its impulse response tends to zero as $t \rightarrow \infty$. Since the transient response resulting from an arbitrary input, or disturbance, can contain only time functions of the type which occur in the impulse function, it is a corollary to this definition that an asymptotically stable linear process will eventually return to equilibrium for any transient disturbance. It can also be shown that the output of an asymptotically stable process will remain bounded if the input is bounded. In the real world, processes are not created to simply be stable for a short period. They must be stable in the long run. Therefore asymptotic stability ($t \rightarrow \infty$) is the prime objective. To shorten notation, in the rest of this work the word stable will signify asymptotically stable. Finally if a system is stable one desires to know how close it is to being unstable. This is the additional concept of relative stability as compared to absolute stability, previously discussed.

B. SURVEY OF THE LITERATURE ON STABILITY:

Numerous criteria based on the previous definitions are available to provide an answer to the question of stability for linear systems. These methods are either analytical or graphical. Basically, they all start with the characteristic equation derived by Laplace transformation. Traditionally, the analytical approach is reflected in the Routh and Hurwitz criteria and the Root-Locus method. The first two answer the question of absolute stability while the last one determines relative stability. On the other hand, names like Nyquist, and Bode often appear when the graphical approach is selected. Numerous other criteria have also been derived.⁴²

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(i) The Analytical Methods:

The Routh and Hurwitz criteria give fast results ⁴³ for the following characteristic equation:

$$a_0 + a_1 s + a_2 s^2 + a_3 s^3 + \dots + a_n s^n = 0 \quad (44)$$

where the a_i 's are constants. This equation corresponds to a process governed by ordinary differential equations with no differences, and where n , the number of roots of equation (44), is finite.

The exponential terms in equation (43a), on the other hand, are periodic by definition:

$$\begin{aligned} e^{-st_1} &= e^{-(\sigma + i\omega)t_1} = e^{-\sigma t_1} \cdot e^{-i\omega t_1} \\ &= e^{-\sigma t_1} \cdot e^{-i(\omega t_1 + 2n\pi)} \quad (n = 0, \pm 1, \pm 2, \dots) \end{aligned} \quad (45)$$

$$(i = \sqrt{-1})$$

and thus equation (43a) is transcendental with an infinite number of roots.

Although the stability requirements are the same for a purely algebraic characteristic equation as well as for one with exponential terms, the finite analytical criteria fail to give exact results for the latter. This is due to the presence of the infinite order polynomial which represents the exponential as follows:

$$e^{-st_1} = 1 - t_1 s + \frac{t_1^2 s^2}{2!} - \frac{t_1^3 s^3}{3!} + \dots \quad (46)$$

Attempts have been made to approximate e^{-st_1} , by truncating this Taylor series expansion. This approximation has been shown to be contradictory. Choksy ⁴⁴ demonstrated that, depending on where one truncates the series, the process can be found to be both stable and unstable when the Routh criterion is later applied, for instance. Other approximations to e^{-st_1} , have been derived in order that the Routh criterion can handle it. Truxal ³¹ discusses various ones in increasing order of sophistication. Firstly, he starts with the exponential function expressed as a limit:

$$e^{-st_1} = \lim_{n \rightarrow \infty} \left(\frac{1}{1 + \frac{t_1 s}{n}} \right)^n \quad (47)$$

but mentions its weakness at $n = 3$. Secondly, he proposes the McLaurin series approximation as:

$$e^{-t_1 s} = \frac{1}{1 + t_1 s + \frac{t_1^2 s^2}{2!} + \frac{t_1^3 s^3}{3!} + \dots} \quad (48)$$

which is derived from the previous Taylor series expansion, again with limitations. Thirdly, the Padé approximation table is presented. This is a ratio of two finite polynomials with selected coefficients (see Figure 5). One can choose any one fraction in this table as an approximation to e^{-s} . The higher order polynomials in the numerators or denominators produce a better accuracy. The Padé approximation and its extension, the technique of Single and Stubbs which produces a similar ratio, are favorite methods in industry.^{32, 35}

Instead of customizing the troublesome exponential to the Routh criterion, an effort has been directed to the root locus technique. Krall^{46, 47} was initially interested in the roots of transcendental equations from an involved mathematical point of view. Finally in 1967, he produced a digital algorithm⁴⁸ used to generate the root locus diagram for an equation having an e^{-s} term. Such studies were pioneered by Chu⁴⁹ in 1952.

All of these works, so far, have considered only the case of the single pure time delay, thus providing restricted results. Contributions to the problem of multiple delays are very rare, probably due to the fact that most real-world applications assume that the multiple case is a trivial extension of the single case. This is not always so, as discussed in Section III. Yuan-Yun, Quing and Lian⁵⁰ produced a paper on multiple delays from a sophisticated analytical view. In addition, Shaughnessy and Kashiwagi⁵¹ have derived a stability indicative function for the multiple delay case.

(ii) The Graphical Approaches:

These methods, such as the Nyquist criterion, can determine exactly the relative stability of systems with pure time delays. They handle the latter without any numerical approximations. The Nyquist criterion and its application to the case of a single delay are outlined in Appendix D. Figures 6a and 6b refer to the text in this appendix.

C. THE SATCHE TECHNIQUE:

It can be seen that the simple example used in Appendix D produces a Nyquist plot difficult to draw. The diagram is even more troublesome to evaluate for higher order polynomials, not to mention the multiple non-trivial cases. Satche¹⁷ derived the following elegant procedure which greatly simplifies the Nyquist plot. One supposes that the characteristic polynomial in equation (43a) can be separated into pure algebraic and pure exponential parts. Therefore, $F(s, e^s)$ can be written as:

$$F(s, e^s) = F_1(s) - F_2(e^s) \quad (49)$$

The loci for F_1 and F_2 are separately drawn in Figure 6c according to the Nyquist method. Looking at that figure, the $F_1(s)$ curve is a closed form algebraic polynomial in s . It closes on itself and part of it resides at an infinite distance from the origin. The area it encloses is cross-hatched. The $F_2(e^s)$ locus, on the other hand, describes a finite contour around the origin. The area enclosed by F_2 is covered with dots. In the Nyquist Diagram, one recalls that it is necessary to determine the number of rotations performed by a vector, originating at the origin, whose head follows the contour Γ as shown in figure 6a. This number of revolutions is identical to the number of roots with positive real parts (unstable roots). In the Satche diagram (Figure 6c), a similar vector is drawn for each contour in F_1 and F_2 . Then, according to equation (49), $F(s, e^s)$ is the resultant vector for $F_1 - F_2$. It joins the points A and B in the diagram. Both ends of the vector \vec{AB} are now free to move. The reader can follow the motion of \vec{AB} as \vec{OA} and \vec{OB} are rotated clockwise in Figure 6c. \vec{AB} revolves on itself while its magnitude changes. At the lower intersection of F_1 and F_2 the magnitude is zero and as soon as the intersection is passed, \vec{AB} emerges with a complete half rotation. The same phenomenon occurs at the other intersection. Thus the intersection points are critical. Each point forces \vec{AB} to rotate half a revolution. The Nyquist criterion can now be modified to impose on \vec{AB} the condition that if the vector performs one or more complete revolution on itself then the system is unstable. In summary, three cases arise in this new criterion called the Satche criterion:

- (1) If the two curves in the Satche diagram are completely disjoint, the system is stable. This case is obvious since the angle of the rotation of \vec{AB} is bounded and always less than 360° .
- (2) If one curve completely encircles the other without intersection, the system is unstable; this is another obvious case, since at least one revolution of \vec{AB} is inevitable.

- (3) If the two curves intersect, then stability can be determined as follows. Recalling that $s = i\omega$ in the Nyquist plot, let $\omega_1 \leq \omega \leq \omega_2$ be the parameter range of two successive outermost intersection points on the curve of F_1 during which F_1 lies within the area bounded by the F_2 curve, (i.e. the area where instability can occur corresponding conformally to the right-half of the complex s -plane in Figure 6b.) Then if that part of the curve of F_2 corresponding to the same ω range lies completely outside the area bounded by F_1 , then the vector \vec{AB} suffers no net rotation and the system is stable.

This case is demonstrated by Figure 6c as follows: One notes that the vector $\vec{A_2B_2}$ just above the first intersection proceeds to become $\vec{A_3B_3}$ without any rotation, thus indicating stability. Once the upper intersection is crossed, $\vec{A_3B_3}$ rotates 180° to become $\vec{A_4B_4}$. The latter proceeds clockwise, and somewhere after \vec{AB} , it eventually has the same direction as $\vec{A_3B_3}$ indicating one complete revolution. Any further rotation dictates instability.

To the author's knowledge the above Satche criterion has been applied only to single pure time delay cases. It is possible to extend the criterion for general characteristic equations with numerous exponentials provided that the algebraic and exponential terms can be separated. The proof of this extension involves a simple axiom from vector algebra, namely the law of uniqueness in vector addition. This states that to every pair of vectors, there is a unique vector, called their sum. As an illustration, equation (49) can be written in detail as:

$$F(s, e^s) = F_1(s) - \sum_{j=2}^i F_j(e^{-st_{j-1}}) \quad (50)$$

where F_2, F_3, \dots, F_i form a series of curves on the Satche diagram, each one corresponding to a different exponential in equation (43a). One can add up the vector joining a point on the F_i locus to a corresponding one on the F_{i-1} locus. The result is a single vector (the addition law) which corresponds to a new locus, $F_i + F_{i-1}$. The new vector is then added to the appropriate vector for F_{i-2} . The procedure is repeated until a final curve and corresponding vector are obtained containing all of the exponentials. This curve is then compared to $F_1(s)$ as done in the simplified Satche diagram, in Figure 6c. For the case of numerous exponentials, the F_2 curve has more than one intersection with F_1 . For stability, one should consider only the outermost crossovers.

D. COMPUTER IMPLEMENTATION

The Generalized Satche criterion is readily adaptable to computer programming. Such effort is performed in order that the computer can determine numerically whether a system is stable or not without recourse to a diagram. The Satche diagram thus becomes secondary and serves only as a graphical summary of the numerical work.

Proper instructions in FORTRAN are set up for an IBM 360/91 system. The program, accepts any number of pure time delays, within the capability of the machine. The graphical capabilities of CALCOMP (Computer Graphics) are then used and the final result is a Satche diagram for a non-trivial multiple pure time delay problem drawn by the machine. The application that follows illustrates the procedure.

SECTION V

APPLICATIONS

A. AN OPERATIONAL MODEL WITH TWO NON-TRIVIAL PURE TIME DELAYS:

Consider a real network possessing both an information and a material, or resource, flow. Information flow is almost instantaneous since it usually involves electrical or electronic means such as telecommunications. Resource flow, on the other hand, suffers delays in schedule such as those in transportation and in office bureaucracy. The process is shown in Figure 7 and consists of:

1. A management sector made up of:
 - (a) A single junction point where feedback information from two destination points is subtracted from the forward resource flow rate entering, (data processing),
 - (b) A control having the decision-making option to vary a factor, K , which, in turn, takes on fractional values between 0 and 1 and multiplies the resource flow rate such as to increase or diminish it.
2. An undesirable forward time lag $(1/[\tau_1 s + 1])$ which deforms the characteristics of the resource rate function. The time-lag occurs quite often in the real world, as demonstrated by Forrester.³⁹
3. An integrator which changes the resource flow rate into a flow of individual units.
4. Two pure time (or transport) delays, hindering the flow which eventually reaches two destination points.
5. An information line connecting each of the two destinations with the original junction, bringing back knowledge on the volume level in time-past due to the presence of the delays.

The input takes the form of a rate (units/time) since the management sector traditionally deals with rates. The destination points on the other hand, situated

at a lower level in the organizational hierarchy, usually report on the volume level of actual units received. An integration is therefore required to transform the rate of flow into a level of flow. Figure 8 represents the mathematical flow graph for Figure 7. One notes that the model constitutes a non-trivial case since the two destination points report to the same centralized data processor. The differential-difference equations extracted from Figure 8 are:

$$\dot{x}_2(t) = \dot{x}_1(t) - b_1 x_6(t) - b_2 x_7(t) \quad (51)$$

$$\dot{x}_3(t) = K \dot{x}_2(t) \quad (52)$$

$$\dot{x}_4(t) = e^{-t/\tau_1} \dot{x}_3(t) \quad (53)$$

$$x_5(t) = \int \dot{x}_4(t) dt \quad (54)$$

$$x_6(t) = x_5(t - t_1) \quad (55)$$

$$x_7(t) = x_6(t - t_2) \quad (56)$$

where

$$(\dot{}) = \frac{d}{dt}$$

$\dot{x}_1(t)$ = an arbitrary input rate (units/time) originating from the environment.

$\dot{x}_2(t)$ = the resultant rate exiting from the junction point (units/time)

$\dot{x}_3(t)$ = the rate of resource flow (units/time) resulting from the management control on $\dot{x}_2(t)$.

$\dot{x}_4(t)$ = the rate of resource flow (unit/time) transformed by the undesirable time-lag.

$x_5(t)$ = the level of resource flow (units), once the above rate (units/time) is integrated.

$x_6(t), x_7(t)$ = the levels (units) at the two destinations during which the flow suffers pure time delays of t_1 and t_2 , respectively.

K = the management control (dimensionless) which is to be varied.

b_1, b_2 = the feedback constants (1/time).

Equation (52) describes the transformation at the central junction; feedback terms such as $b_1 x_6(t)$ and $b_2 x_7(t)$ (units/time) represent the rate at which the resource units entering the destination points are being rejected due to overflow created by constraints. A linear relationship has been assumed between the rejection rates $b_1 x_6(t)$ and $b_2 x_7(t)$ and the levels $x_6(t)$ and $x_7(t)$, since a higher "crowding" of units at these receiving stations causes the latter to produce a higher rejection rate. The central junction processor (such as a computer) is programmed to take the above into account and thus pre-plan the schedule by subtracting the rejection rates from the entering resource rate as shown in equation (52). That difference enters the management control where an optimal value of K has to be determined. The latter should allow for maximum entering flow rate and, at the same time, provide asymptotically stable level fluctuations at the destination points.

B. APPLICATION OF THE GENERALIZED SATCHE TECHNIQUE:

To determine such optimum value of K , the generalized Satche technique is used. Applying first the Laplace Transform to equations (51-56), six equations are obtained in the s domain. Combining these into one and rearranging terms as in equation (5), the characteristic equation becomes:

$$F(s, e^s) = s(\tau_1 s + 1) + b_1 K e^{-st_1} + b_2 K e^{-s(t_1+t_2)} = 0 \quad (57)$$

The same result can be obtained directly from the graph in Figure 8 along with the general equation (43a) for $N = 2$. The latter becomes:

$$\begin{aligned} F(s, e^s) &= 1 + G_1 H_1 D_1 + G_2 H_1 D_1 H_2 D_2 \\ &= 0 \end{aligned} \quad (58)$$

and since in Figure 8,

$$G_1 = -b_1$$

$$G_2 = -b_2$$

$$H_1 = K \cdot \frac{1}{\tau_1 s + 1} \cdot \frac{1}{s}$$

$$H_2 = \text{unity}$$

$$D_1 = e^{-st_1}$$

$$D_2 = e^{-st_2}$$

then equation (58) eventually becomes equation (57). Since equation (57) takes the form of equation (50), one has:

$$F(s, e^s) = F_1 - F_2 \quad (59)$$

with

$$F_1 = s(\tau_1 s + 1) \quad (60)$$

and

$$F_2 = -b_1 K e^{-st_1} - b_2 K e^{-s(t_1+t_2)} \quad (61)$$

Substituting $s = i\omega$ (see Appendix D) into equations (60) and (61) one obtains ($i = \sqrt{-1}$):

$$F_1(\omega) = -\tau_1 \omega^2 + i\omega \quad (62)$$

$$\begin{aligned}
F_2(\omega) &= -b_1 K e^{-i\omega t_1} - b_2 K e^{-i\omega(t_1+t_2)} \\
&= -b_1 K \{\cos(\omega t_1) - i \sin(\omega t_1)\} \\
&\quad - b_2 K \{\cos[(t_1 + t_2) \cdot \omega] - i \sin[(t_1 + t_2) \cdot \omega]\} \quad (63)
\end{aligned}$$

Separating real and imaginary parts yields:

$$X_1(\omega) = \text{Re } F_1(\omega) = -\tau_1 \omega^2 \quad (64)$$

$$Y_1(\omega) = \text{Im } F_1(\omega) = \omega \quad (65)$$

$$X_2(\omega) = \text{Re } F_2(\omega) = -b_1 K \cos(\omega t_1) - b_2 K \cos[(t_1 + t_2) \cdot \omega] \quad (66)$$

$$Y_2(\omega) = \text{Im } F_2(\omega) = b_1 K \sin(\omega t_1) + b_2 K \sin[(t_1 + t_2) \cdot \omega] \quad (67)$$

where Re = "real part of" and Im = "imaginary part of." The constants in equations (64-67) are arbitrarily selected as:

$$b_1 = b_2 = 1 \quad (68, 69)$$

$$t_1 = 1 \quad (70)$$

$$t_2 = 2 \quad (71)$$

$$\tau_1 = 1.0 \quad (72)$$

$$K = 0.45 \quad (73)$$

remembering that $0 \leq K \leq 1$.

C. COMPUTER APPLICATION OF SECTION (B)

The general program described in Appendix E is utilized to obtain the necessary numerical and graphical results for stability. The procedure consists of the following five major steps.

(i) Generation of Values for X_1 , Y_1 , X_2 , Y_2 in Equations (64-67):

The parameter ω is varied over an interval of $-\pi$ to $+\pi$ with an incremental value arbitrarily chosen as $\pi/256$. Thus ω takes on 513 values. Corresponding numerical values for X_1 , Y_1 , X_2 , and Y_2 are generated by the computer utilizing equations (64-67). The output is shown in Figure 9. There, the top headings state the numerical values of the constants as in equations (68-73). "B(1) TO B(M)" and "T(1) TO T(M)" represents equations (68-71). Subscripts appear in parantheses, with $M = 2$, the number of pure time delays in this case. Due to the presence of sine and cosine in equations (66-67), X_2 and Y_2 are periodic and thus redundant outside the $+\pi$ to $-\pi$ interval in ω . The increment for ω is selected such as to produce satisfactory smooth curves when computer graphics are later used.

The nature of equations (64-65) dictates that F_1 is a parabola symmetric about x-axis, open to the left and going through the origin on the Satche diagram. It closes on itself at infinity as in Figure 6c. The area of the right of this parabola is enclosed by F_1 . Equations (66-67) indicate that F_2 is a closed curve around the origin and thus F_1 and F_2 intersect once or more in the Satche diagram. Therefore, of the three possible cases outlined in the Satche criterion in Section IV, only the last and most general one will be examined here.

(ii) Locating the Regions of Intersection of F_1 and F_2 :

The set X_1 and Y_1 along with the set X_2 and Y_2 each corresponds to an ordered pair (a point) on the F_1 and F_2 curves, respectively. It is first necessary to determine numerically the points where these two curves intersect. The SCAN SUBROUTINE, a numerical algorithm (see Appendix E), performs this required step as follows. Starting with the first line of X_1 and Y_1 in Figure 9, the entire ordered X_2 , Y_2 set is scanned for that pair on F_2 closest in distance to that point (X_1 , Y_1) on F_1 . The algorithm utilizes the distance definition for two ordered pairs:

$$D = \sqrt{(X_1 - X_2)^2 + (Y_1 - Y_2)^2} \quad (74)$$

The pair X_2, Y_2 found closest is at a distance identified as DMIN. Furthermore, if DMIN is less than or equal to the value 0.025 (an estimated very small term) then X_2, Y_2 is not only the closest to X_1, Y_1 but at these four values DMIN is near zero and F_1 and F_2 intersect. The next pair (X_1, Y_1) on F_1 in Figure 9 is selected and the above procedure is repeated until all of the points on F_1 have been examined. Figure 10 gives the results of the work done by such algorithm. The code $N = 0$ signifies that $X_2 = \text{Re} F_2$ and $Y_2 = \text{Im} F_2$, on that line, form the closest non-intersection point resulting from the scanning of the entire set of X_2, Y_2 . Furthermore, $N = 1$ indicates a region of intersection. One notes that F_1 and F_2 do not have each the same ω at $N = 0$ or 1. The examination of the column for N in Figure 10 reveals that a long series of 0's first appear followed by a few 1's, and then more 0's. These 1's indicate a region of intersection, with probably the middle line, within these 1's, as being the desired point. The fact that a region and not a point of intersection can be determined is explained by the small value 0.025 taken as criterion. The smaller it is, the smaller the 1's region becomes and the better the resolution. However, an estimated value that is too small, such as less than 0.025, can cause the scanning algorithm to miss some intersections depending on how close to orthogonal (perpendicular) the meeting of F_1 is to F_2 .

(iii) Locating the Outermost Intersections:

The computer program next examines which two regions of intersections are the outermost ones with an increasing ω for F_1 . These two points are identified in Figure 11. The last column in that figure gives the bounds for ω corresponding to F_2 . The numerical values under "OMEGA OF F1" in that figure correspond to the lower and upper bound on ω for F_2 and refer to ω_1 and ω_2 , respectively, in the Satche criterion.

(iv) Determining if the Values of $F_2 (\omega_1)$ through $F_2 (\omega_2)$ are Enclosed by F_1 :

The set of pairs $X_2 (\omega_1), Y_2 (\omega_1)$ through $X_2 (\omega_2), Y_2 (\omega_2)$ represent the critical part of F_2 and must be examined for possible intersection with F_1 . Subroutine SCAN² is again utilized, but this time F_1 and F_2 are replacing each other in the algorithm, namely, the entire F_1 is scanned for each selected point X_2, Y_2 within $F_2 (\omega_1)$ to $F_2 (\omega_2)$. If an intersection occurs then a section of that critical part of F_2 must be to the right of F_1 and instability is dictated. If no intersection occurs then two possible cases arise, either $F_2 (\omega_1)$ through $F_2 (\omega_2)$ is completely to the right or completely to the left of F_1 . If it is to the right, the system is unstable, if to the left, stability is declared, according to the Satche criterion. The computer program performs the above logic and produces output of this step in Figure 12. One should read the explanatory N and L codes in the figure.

(v) The Satche Diagram:

The program next stores the preceding results (Figures 9 through 12) on a magnetic tape. Subroutine SATCHE performs this step such that the tape can be later mounted on a Calcomp (California Computing) System. This hardware electronically deciphers the code and mechanically draws the Satche diagram. The latter is shown in Figure 13 and summarizes the complete effort for the case of $K = 0.45$. One can observe five intersections between the parabola for F_1 , identified with +++, and F_2 . F_1 closes on itself at infinity in the far right plane. That part is not drawn since it is of secondary importance. The critical part of F_2 (ω_1) through F_2 (ω_2), identified by **** goes to the right beyond the outermost intersections and therefore the case is unstable. Figure 14 shows a highly unstable case with $K = 1.0$. In this instance the resource flow rate is at a maximum with no management restraint. Figure 15 indicates a stable system for $K = 0.25$. Figure 16 shows a case where for $K = 0.35$ the critical part of F_2 (ω_1) through F_2 (ω_2) is to the left of F_1 and terminates at the outer intersection points. The value $K = 0.35$ is therefore an optimal case since it is the largest K for which the process is still stable.

It is possible to confirm the integrity of the generalized Satche criterion, the computer program and the plotting routine by performing a relatively fast digital simulation. Since this approach falls outside of the stability theory presented so far it becomes a fairly rigorous test. The Continuous System Modeling Program (CSMP) package provided by IBM⁵², performs such simulation. Figure 17 shows the CSMP model which corresponds to the flow graph in Figure 8. The simulation produces the time response of the process when an input is applied. It solves the system of differential-difference equations (51-56) utilizing a digital integration method, the Adams technique, provides a discretized evaluation of the two pure time delays, and finally produces a numerical and graphical display of the output to the system. In this model simulation, a unit step function is arbitrarily chosen as input. One is interested in the fluctuations as time progresses of x_7 , the final destination level, representing the output. Figure 18 shows the results for $K = 0.45$. First of all the response of x_7 does not appear until time = 3.00. This represents the overall initial pure time delay of $t_1 (= 1) + t_2 (= 2) = 3$. Furthermore the level at x_7 in Figure 18 shows growing oscillations indicating an unbounded output thus causing the overall process to be unstable. Figure 19 shows a stable condition when the above simulation is repeated with $K = 0.25$, this time. Here, the final destination level fluctuates significantly at the start but such transient behavior eventually decays to a settled level as dictated by the control. The stability for this process is evident at the end of 80 time units. The simulation is performed a final time with $K = 0.35$. The oscillations for this case are shown in Figure 20. The oscillation heights slightly decrease from one peak to the other revealing eventual asymptotic stability but with a "settling" time that is much longer than for $K = 0.25$. The decision-making control is confronted with two alternatives. Depending on the trade-off objective,

the management can apply a tight control on the resource flow such as $K = 0.25$ and obtain a fast settling but low, level at the destination; or else liberally increase K to the allowed maximum value of 0.35 for stability, causing level fluctuations which take a longer time to settle but obtaining a higher final level. In summary, the graph in Figure 18 reveals an unstable system, for $K = 0.45$. For $K = 0.25$, one has a definite stable case. And for $K = 0.35$, the system, though stable, is very close to be uncontrollable. The simulation is therefore in complete agreement with the predictions of the Satche technique applied in the two-delay case.

D. OTHER NON-TRIVIAL PURE TIME DELAY CASES:

Since the computer program is a general one, numerous delay cases can be examined.

(i) Single Pure Time Delay:

This situation is the simplified version with no time lag illustrated in Appendix D, (Equation D-5). Figures 21 through 23 present three cases for $K = 2, 1, \pi/2$ and which are unstable and optimal, respectively. In the last case $\pi/2$ has been approximated by 1.57. The Satche functions $F_1(\omega)$ and $F_2(\omega)$ for equation (D-5) are $i\omega$, a straight line through the imaginary axis and $-Ke^{-i\omega}$, a circle of radius K . Furthermore, the fact that F_2 in Figures 21 and 23 looks like an ellipse rather than a circle is no surprise since it is simply due to the graphic subroutine trying to optimize the area where the diagram is to be drawn; numerically, Figures 21 and 23 represent a circle. Figures 21 through 23 are in agreement with the results in Refs. 42 and 43.

Figures 24 and 25 show an unstable and a stable single pure time delay case with time lag, respectively. The characteristic equation for these two cases is taken as equation (57) with $b_2 = 0$. These two figures illustrate, somewhat, how to stabilize a process by manipulating K and τ_1 . A decrease in τ_1 in Figure 25 flattens to the right the parabola of Fig. 24 as dictated by equations (64-65). This allows a larger region of F_1 to the left of F_2 to be exposed and on which the critical part of F_2 (***) can possibly fall, thus producing a stable case. Another manipulation involves a decrease in K thus concentrating the critical part of F_2 (***) to the left of F_1 , as seen in going from Figure 24 to 25. Here, one also notes that a decrease in the time delay improves the stability.

(ii) Extension to Cases with Three, Four and Five non-trivial Pure Time Delays:

The preceding model is extended to incorporate up to five pure time delays. The general equation (43a) and Figure 4d are utilized with $N = 3, 4$, and 5 . The characteristic equation for three delays is an extension of equation (57). It is given as:

$$F(s, e^s) = s(\tau_1 s + 1) + b_1 K e^{-st_1} + b_2 K e^{-s(t_1+t_2)} + b_3 K e^{-s(t_1+t_2+t_3)} = 0 \quad (75)$$

with

$$F_1(s) = s(\tau_1 s + 1) \quad (76)$$

and

$$F_2(e^s) = -[b_1 K e^{-st_1} + b_2 K e^{-s(t_1+t_2)} + b_3 K e^{-s(t_1+t_2+t_3)}] \quad (77)$$

For four delays equation (57) becomes:

$$\begin{aligned} F(s, e^s) = & s(\tau_1 s + 1) + b_1 K e^{-st_1} \\ & + b_2 K e^{-s(t_1+t_2)} + b_3 K e^{-s(t_1+t_2+t_3)} \\ & + b_4 K e^{-s(t_1+t_2+t_3+t_4)} = 0 \end{aligned} \quad (78)$$

with

$$F_1(s) = s(\tau_1 s + 1) \quad (79)$$

and

$$\begin{aligned} F_2(e^s) = & -[b_1 K e^{-st_1} + b_2 K e^{-s(t_1+t_2)} \\ & + b_3 K e^{-s(t_1+t_2+t_3+t_4)}] \end{aligned} \quad (80)$$

And for five delays, one has:

$$\begin{aligned}
 F(s, e^s) = & s(\tau_1 s + 1) + b_1 Ke^{-st_1} + b_2 Ke^{-s(t_1+t_2)} \\
 & + b_3 Ke^{-s(t_1+t_2+t_3)} + b_4 Ke^{-s(t_1+t_2+t_3+t_4)} \\
 & + b_5 Ke^{-s(t_1+t_2+t_3+t_4+t_5)}]
 \end{aligned} \tag{81}$$

with

$$F_1(s) = s(\tau_1 s + 1) \tag{82}$$

and

$$\begin{aligned}
 F_2(e^s) = & - [b_1 Ke^{-st_1} + b_2 Ke^{-s(t_1+t_2)} + b_3 Ke^{-s(t_1+t_2+t_3)} \\
 & + b_4 Ke^{-s(t_1+t_2+t_3+t_4)} + b_5 Ke^{-s(t_1+t_2+t_3+t_4+t_5)}]
 \end{aligned} \tag{83}$$

Figures 26 through 28 illustrate near optimum cases (optimal K) for the three-pure time delay problem. Figures 29, 30, 31 and 32 presents diagrams for the case of four and five pure time delays, respectively. Figures 29 and 30 shows a stable situation while Figure 31 indicates instability. Figure 32 gives a stable case for five pure time delays.

SECTION VI

CONCLUSION

A. JUSTIFICATION OF THE DIFFERENTIAL-DIFFERENCE EQUATION:

The models forged so far, containing a combination of continuous and discrete equations (with derivatives and differences) can be simplified to a purely discrete form involving only difference equations. Such a discrete model poses no difficulties in the stability analysis. In this case, the characteristic polynomials involve the Z transform which acts efficiently on transcendental terms.⁴² As seen, this is not so in the case of the continuous-discrete formulation. In justification of the continuous approach taken here, the adjectives discrete and continuous are relative terms depending on the application. As an illustration, feedback information flow entering the decision-making headquarters in the form of a telephone call once a day, for instance, appears as a discrete event among other daily office activities. However, if these daily phone reports serve to inform the management on a transportation network suffering time delays of monthly magnitude, then they form a continuous stream while the transportation flow is of much more discrete nature.

B. COMPUTER TIME:

The total computer time for analyzing the stability of the models presented so far requires at most 30 seconds of processing per case. Therefore the contemplation of developing more efficient routines is not pursued.

C. THE CHOICE BETWEEN THE LAPLACE APPROACH AND THE TIME DOMAIN APPROACH:

The Laplace transform method used here differs noticeably from the theory developed in the time domain, often called the state variable theory. The first one is rigorous since it optimizes a model by insuring its stability. The second approach, instead, heuristically optimizes an objective function. This function is often left at the analyst discretion to formulate, based on what he feels are important variables.

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D. SUMMARY

R. Bellman⁵³ in a survey on the literature related to Kalecki's Model²², discusses two characteristic equations, one of the form:

$$\lambda s^2 - s + b - ce^{-s} = 0 \quad (84)$$

where λ , b and c are constants, and the other one as:

$$s = b(e^{-st_1} - e^{-st_2}) \quad (85)$$

where b is a constant and $t_2 = t_1 + 1$. The above author comments that the determination of the roots of eq. (84), the one containing a simple delay, has been fully examined for stability. Equation (85) on the other hand, containing two coupled delays is, quoting the author, "considerably more difficult" and where "the details are much more perplexing" in the analytical stability approach. The extended Satche method developed herein, can easily determine the stability of eq. (85) from the graphical point of view, thus by-passing the analytical determination of the roots. In addition, since this technique is computer-based, it serves as a fast and accurate method for a particular user.

Furthermore, the extension to multiple delays is quite impossible without a computer, justifying the use of a machine. The complexity of Figure 31 supports such statement.

One notes that, in general, as the number of pure time delays increases, the allowable value for K decreases allowing for less resource flow. The undesirability of delays in the real world is thus mathematically illustrated. Most important is the total agreement between the simulation and the criterion prediction. This Satche extension must undergo other tests if it is to become an established technique. Finally, the validity of the model for multiple inputs and outputs and for non-deterministic random variables remains to be demonstrated, thus providing grounds for further research.

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APPENDIX A

BASIC DEFINITIONS USED IN THIS WORK

1. A system or process: an arrangement of elements connected or related in such a manner as to form a working entirety.
2. A control system: a system able to regulate or command itself or other systems.
3. A feedback control system: a control system where the output of some controlled system variable is compared with the input to the system so that the appropriate control action may be formed as a function of the output or input. Feedback increases the accuracy of a control system and favors overall system stability.
4. Modeling: a mathematical method used to express the real world through:
 - (a) Functional or differential equations.
 - (b) Block diagrams, similar to flow charting.
 - (c) Flow graphs.

Block diagrams and flow graphs are schematic representations of the model's equations and real components. Appendix B explains the block diagram notation. The flow graph method appears in Appendix C.

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APPENDIX B

TERMINOLOGY OF FEEDBACK AND CONTROL: THE BLOCK DIAGRAM AND THE TRANSFER FUNCTION

A. THE BLOCK DIAGRAM:

The block diagram notation is a graphical representation of the overall cause and effect relationship between input and output of a process or system. The components or elements of a system is each characterized by a box (or block) with an arrow penetrating and exiting, representing an input and output, respectively. The box usually contains a mathematical transformation such as a differentiation, integration, division or multiplication which acts on the input to produce the desired output. Figure 1a illustrates such a representation. All the blocks are related to one another and eventually form the overall system. Figure 1b illustrates a generalized beedback control system. In this diagram, the control block function can be varied, the process is fixed and the feedback is subtracted (negative feedback) or added (positive feedback) from an input originating outside the system.

B. THE TRANSFER FUNCTION (T.F.):

The ratio of the output to the input of a system constitutes the transfer function. This ratio can be determined analytically by equating the output to the transformed input for each block of the system. This gives a series of coupled equations which can be combined into a single equation containing the original input and final output. The ratio of these two is then readily obtained.

The method in Appendix C provides a more elegant and rigorous way of determining the transfer function for linear, deterministic systems.

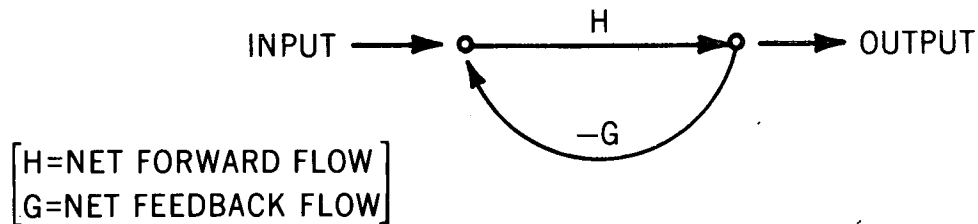
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APPENDIX C

NETWORK FLOW GRAPHS AND MASON'S LOOP RULE

A. NETWORK OR SIGNAL FLOW GRAPHS:

Basically, the block diagram notation in Figure 1b can be simplified to an algebraic graph as follows:



where the transfer function is

$$\frac{\text{OUTPUT}}{\text{INPUT}} = \text{T. F.} = \frac{GH}{1 - GH} \quad (\text{C-1})$$

The arrows in the above diagram are called branches while the destination dots are called nodes. A path is a branch or a continuous sequence of branches which can be traversed from one signal (node) to another signal (node). A loop is a closed path which originates and terminates on the same node and, along which, no node is met twice.

B. MASON'S LOOP RULE:

Equation (C-1) is obtained by equating the incoming flow of values to the outgoing one at each node, since no accumulation can occur in a flow graph. This produces a set of equations which reduce to equation (C-1). A shorter way to derive the transfer function is to utilize Mason's Loop Rule (see ref. 43). The Rule states that, given a flow graph, the transfer function between two nodes, i and j respectively, is given by:

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$$T. F._{ij} = \frac{\sum_k P_{ijk} \Delta_{ijk}}{\Delta} \quad (C-2)$$

where

P_{ijk} = k^{th} forward path from variable nodes i to j ,

Δ = determinant of the graph,

Δ_{ijk} = cofactor of the path P_{ijk}

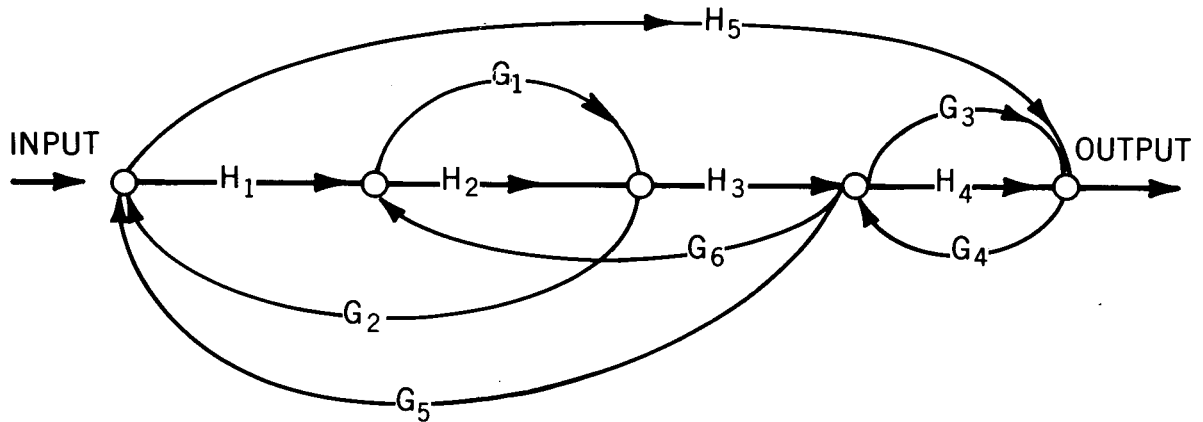
and the summation is taken over all possible k paths from the i to j nodes. The cofactor Δ_{ijk} is the determinant with the loops touching the k^{th} path removed. The determinant Δ is:

$$\Delta = 1 - \sum_{n=1}^N L_n + \sum_{m=1, q=1}^{M, Q} L_m L_q - \sum L_n L_s L_t + \dots \quad (C-3)$$

where L_q equals the value of the q^{th} loop transmittance. Therefore, in words, equation (C-3) signifies that $\Delta = 1 -$ (sum of all different loop branches) + (sum of branch products of all combinations of two non-touching loops) - (sum of the branch products of all combinations of three non-touching loops) + Two loops are non-touching if they do not have a common node.

C. ILLUSTRATION OF MASON'S LOOP RULE:

Consider the following flow graph:



There are five forward paths connecting the input to the output:

$$P_1 = H_1 H_2 H_3 H_4 \quad (C-3)$$

$$P_2 = H_1 G_1 H_3 G_3 \quad (C-4)$$

$$P_3 = H_1 G_1 H_3 H_4 \quad (C-5)$$

$$P_4 = H_1 H_2 H_3 G_3 \quad (C-6)$$

$$P_5 = H_5 \quad (C-7)$$

There are four self loops:

$$L_1 = H_1 H_2 G_2 \quad (C-8)$$

$$L_2 = H_4 G_4 \quad (C-9)$$

$$L_3 = H_1 H_2 H_3 G_5 \quad (C-10)$$

$$L_4 = H_2 H_3 G_6 \quad (C-11)$$

Loops L_1 and L_2 do not touch. Therefore, the determinant is:

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4) + (L_1 L_2) \quad (C-12)$$

The cofactor along P_1 is obtained by removing from Δ the loops that touch P_1 :

$$\Delta_1 = 1 - 0 = 1 \quad (C-13)$$

The same applies for P_2 , P_3 , P_4 and P_5 :

$$\Delta_2 = 1 - 0 = 1 \quad (C-14)$$

$$\Delta_3 = 1 - 0 = 1 \quad (C-15)$$

$$\Delta_4 = 1 - 0 = 1 \quad (C-16)$$

$$\Delta_5 = 1 - L_4 \quad (C-17)$$

and therefore,

$$\begin{aligned} \text{T. F.} &= \frac{P_1 \Delta_1 + P_2 \Delta_2 + P_3 \Delta_3 + P_4 \Delta_4 + P_5 \Delta_5}{\Delta} \\ &= \frac{(H_1 H_2 H_3 H_4) + H_1 G_1 H_3 G_3 + H_1 G_1 H_3 H_4 + H_5 (1 - H_2 H_3 G_6)}{1 - (H_1 H_2 G_2 + H_4 G_4 + H_1 H_2 H_3 G_5 + H_2 H_3 G_6) + H_1 H_2 H_4 G_2 G_4} \end{aligned} \quad (C-18)$$

APPENDIX D

THE NYQUIST CRITERION

A. CONFORMAL MAPPING, THE ENCIRCLEMENT THEOREM:

If the variable s , in the Laplace domain, assumes real and imaginary parts as follows:

$$s = \sigma + i \omega \quad (D-1)$$

(where $i = \sqrt{-1}$), then the characteristic equation in s also assumes real and imaginary parts given by:

$$F(s, e^s) = U(\sigma, \omega) + i V(\sigma, \omega) \quad (D-2)$$

According to the theory of complex variables, a conformal mapping from s to F is defined as follows. To any point (σ, ω) in the s -plane, there corresponds a unique point (U, V) in the F -plane. If the point (σ, ω) traverses a closed contour C in s then (U, V) will trace a corresponding contour, Γ . Figure 6a illustrates such mapping. Accordingly, the Encirclement Theorem states that when C is traversed once clockwise by a point (σ, ω) then the corresponding mapped point (U, V) revolves N times counter-clockwise on Γ in Figure 6a, such that,

$$N = -Z \quad (D-3)$$

where Z is the number of those roots of equation (D-2) which lie within C provided that none of them lie on C itself. A negative N indicates that Γ is traversed in clockwise fashion, while a positive N indicates a counter-clockwise direction. A complete proof is available in Reference 42.

B. THE CONDITION ON THE s-PLANE FOR ASYMPTOTIC STABILITY

The condition for asymptotic stability, exposed in Section IVA, requires that all the roots of the characteristic equation to have negative real parts and therefore to lie in the left-half of the s-plane in Figure 6a. This is briefly explained as follows. The response of a linear stable system to an impulse function must tend to zero as $t \rightarrow \infty$, as stated in Section IVA. This response is generalized as:

$$y(t) = A_1 e^{(\sigma_1 + i\omega_1)t} + A_2 e^{(\sigma_2 + i\omega_2)t} + \dots + A_n e^{(\sigma_n + i\omega_n)t} \quad (D-4)$$

$(i = \sqrt{-1})$

where A_1, A_2, \dots, A_n are constants and $(\sigma_1 + i\omega_1), (\sigma_2 + i\omega_2), \dots, (\sigma_n + i\omega_n)$ are the complex roots of the characteristic equation (D-2). It can be seen from equation (D-3) that if the σ 's are negative, then $y(t)$ decays exponentially thus satisfying the requirement for asymptotic stability. Abiding to such prerequisite, one cancels all possibilities for unstable roots to lie in the left-hand plane. This leaves only the ω -axis and the right-hand s-plane in Figure 6a to examine for unstable roots. The general contour C then specializes to a new contour, C', which is shown in the left portion of Figure 6b. It consists of the whole imaginary axis between $\omega = -\infty$ to $\omega = +\infty$, along with the infinite semi-circular arc encompassing an area to the right of the direction of the arrow, namely, the complete, bounded right-hand plane.

The conformal mapping of this semi-circular closed contour C' onto the F-plane produces a closed Γ contour. First, one considers the semi-circular arc portion (excluding the ω axis) of C' on which s can be written in terms of polar coordinates as:

$$s = R e^{i\theta} \quad (\pi/2 > \theta > -\pi/2) \quad (D-5)$$

where $R = \text{infinity}$, $i = \sqrt{-1}$. The mapping of this portion onto the F-plane generates a clockwise Γ portion which also bounds the entire right hand F-plane with the imaginary F axis. The equation of this Γ portion is immaterial since every point on it is at an infinite distance from the origin. It can be taken as a semi-circle with infinite radius. It is referred to as part of the F-locus at infinity. Returning to the s-plane, the imaginary s-axis remains to be examined from $s = \omega = -\infty$ to $s = \omega = +\infty$. Depending on F, the mapping produces various loci which then connect with the part of the F-locus at infinity to form a closed contour. The area enclosed depends again on the direction of the arrows on Γ .

C. THE NYQUIST CRITERION:

The Criterion basically combines the requirements of section (A) with those of section (B) by determining the number of roots which reside inside C' of the s -plane in Figure 6b, (i.e. those roots with positive real parts and therefore those which cause instability). It can be stated as follows:

According to the Encirclement Theorem, the number of times, N , that the F -locus encircles the origin in a clockwise sense corresponds to the number of roots within C' in the s -plane, i.e. those roots with positive real parts and which cause the system to be an unstable one, according to the stability condition on the s -plane. Therefore, if the F -locus never encircles the origin completely, then the system is (asymptotically) stable, and unstable otherwise.

D. ILLUSTRATION:

The previous criterion is best illustrated as follows. Consider the characteristic equation containing one pure time delay as:

$$F(s, e^s) = s + K e^{-s} = 0 \quad (D-6)$$

where K is an arbitrary constant.

The mapping of s onto $F(s, e^s)$ consists of two parts. First, the semi-circular portion of C' in the s -plane is represented by Equation (D-4). As previously stated, the $F(s, e^s)$ contour corresponding to every point (σ, ω) on that section is itself a semi-circular curve traced at an infinite distance from the origin. It is shown in the F -plane drawn in the right portion of Figure 6b and labeled $|F(s, e^s)| = \infty$. The second part of the mapping traces $F(i\omega, e^{i\omega})$ for ω varying from $-\infty$ to $+\infty$. This tracing is done by firstly substituting $i\omega$ for s in equation (D-5). The result is

$$i\omega + K e^{-i\omega} = 0 \quad (D-7)$$

Since $e^{-i\omega} = \cos \omega - i \sin \omega$, one separates the real and imaginary parts of equation (D-6) to yield:

$$X_1(\omega) = \text{Real } F = K [\cos \omega] \quad (\text{D-8})$$

$$Y_1(\omega) = \text{Imaginary } F = K [i(\omega - \sin \omega)] \quad (\text{D-9})$$

For every ω on the imaginary axis of the s -plane, there corresponds an ordered pair $(X_1(\omega), Y_1(\omega))$ on the F -plane. Numerical values are calculated for X_1 and Y_1 for a varying ω . The result is the mapping of ω onto F . It is shown as the "wiggle-like" portion along the imaginary axis of the F -contour in the right-hand portion of Figure 6b. One notes that three curves appear each corresponding to a different value of K , namely 1.0, $\pi/2$, and 2.0, respectively. All three curves connect with the semi-circular portion of F labeled $|F(s, e^s)| = \infty$ at infinity. The mapping, called the Nyquist plot, therefore completely encloses an area to the right of the direction of the arrows (shaded in Figure 6b) in F .

It can be seen that for $K = 2$ the F -locus encloses the origin twice in the clockwise sense, once very close to it and another time very far from it at $R = \infty$ so that $N = -2$ and therefore $Z = -N = 2$, indicating two roots with positive real parts in C' . The cases for $K = 1.0$ and $\pi/2$ on the other hand reveal no encirclement and therefore constitute stable cases. The value $K = \pi/2$ is at the threshold of stability. It represents the largest allowed value for stability.

APPENDIX E

LISTING OF THE COMPUTER PROGRAM DEVELOPED TO IMPLEMENT THE SATCHE CRITERION FOR MULTIPLE DELAYS AND TO PERFORM COMPUTER GRAPHICS

A. THE SUBROUTINES USED:

The following program is written in FORTRAN IV. The implementation utilized an IBM 360/91 operating under Release 20, a relatively high speed, large memory machine.

Subroutine SCAN determines the intersection points while subroutine SATCHE converts the numerical results into a special code which is transferred on a 9-track magnetic tape. This tape is then mounted on a CALCOMP 780 (California Computing) System which in turn draws the desired plots on a drum using ink and paper.

The subroutines PLOTS, PLOT, LINE, NUMBER, SCALE, and AXIS are made available by the CALCOMP company and assist in drawing the plots according to a desired scale, format and size. CALCOMP also provides a repertory of symbols, letters, and other characters which label the diagram.

B. LIST OF COMPUTER SYMBOLS IN CONTEXT WITH THE MAIN WORK (IN APPROXIMATE ORDER OF APPEARANCE):

$$X1 = X_1 (\omega) = \text{Re } F_1$$

$$X2 = X_2 (\omega) = \text{Re } F_2$$

$$Y1 = Y_1 (\omega) = \text{Im } F_1$$

$$Y2 = Y_2 (\omega) = \text{Im } F_2$$

BUFFER = A dummy vector used as a "scratch pad" by subroutine SATCHE.

OMEGA	= a vector containing all the values for ω .
B	= a vector containing the feedback factors, b_i , in order of appearance in the network forward flow.
T	= a vector containing the exponents of the delays in order of appearance in the network forward flow.
PI	= π
PIB	= selected maximum value for ω .
DPIB	= selected increment value for ω .
I	= subscript for OMEGA, XI Y1, X2, Y2 initially set at 1 and finally set at 513.
NFMAX	= selected maximum number of computer runs per batch. (one run per delay case)
NFILE	= file number assigned to each case, initially set at 1 and finally set at NFMAX.
M	= number of pure time delays in the particular problem being processed.
XK	= K = management control
TAU1	= τ_1 = time lag
MM	= dummy subscript
N1, N2, N3, N1MAX, N2MAX, N3MAX	= various pointers of secondary importance used to produce a convenient output format.
W	= an intermediary value for OMEGA.
XDPI	= an increment smaller than DPIB (of secondary importance)
XT	= a temporary value for T
XX2	= a temporary value for X2 initially set at 0
YY2	= a temporary value for Y2 initially set at 0

K = a dummy subscript
 TEMPX2 = another temporary value for X2
 TEMPY2 = another temporary value for Y2
 WP = a value used to transform ω into a fractional part of π to become readable in the output printout.
 N = a code indicating "intersection" or "no intersection."
 IMAX = 513 = maximum number of values for X1, Y1 and of OMEGA used to evaluate X1 and Y1.
 JMAX = 513 = maximum number of values for X2, Y2 and of OMEGA used to evaluate X2 and Y2.
 IIMIN = subscript of the OMEGA corresponding to ω_1 , the smaller value at which one of the outermost intersections occurs.
 IIMAX = subscript of the OMEGA corresponding to ω_2 the larger value at which the other outermost intersection occurs.
 J = temporary subscript
 ID = the number of values of OMEGA for F1, X1 and Y1 on F1 between the outermost intersections.
 IIIMIN = subscript of a reconstructed OMEGA for F1, X1 and Y1 with the subscript ID = 1 to ID = IIMAX - IIMIN. This is necessary so that the curve F1 and F2 have the same order of magnitude when the Satche diagram is performed.
 SUBROUTINE SCAN: (additional symbols to the above)
 D = distance between (X1, Y1) and (X2, Y2).
 DMIN = smallest distance between (X1, Y1) and (X2, Y2).
 JX = subscript identifying the point (X2, Y2) closest to the given point (X1, Y1).

SUBROUTINE SATCHE: (additional symbols to the above)

XL, YL = values specifying the paper area on which the Satche diagram is to be drawn.

XSUBX, YSUBY, YSUBX, XSUBY = CALCULATED origin of the Satche diagram. (Optimized values such that the diagram is centered within the area specified by XL, YL).

D = a temporary value used in the above optimization.

JJ, NN = dummy subscripts.

XN = a value used for proper format (of secondary importance)

```

C
C   NYQUIST-SATCHE DIAGRAM
C   DIMENSION X1(600), X2(600), Y1(600), Y2(600), BUFFER(7500), OMEGA(600),
C   IB(10), T(10)
C
C   SETTING UP A WORK AREA FOR PLOTTING
C   CALL PLOTS(BUFFER, 7500, 0)
C
C
C   CONSTANTS DEFINED
C   PI=3.14159
C   PIB=PI
C   DPI=16.*16.
C
C   READING THE NUMBER OF COMPUTER RUNS DESIRED
C   READ (5,77) NFMAX
C   77 FORMAT(12)
C
C
C   DO 888 NFILE=1, NFMAX
C   READING IN ORDER THE DESIRED NUMBER OF RUNS,
C   FEEDBACK MULTIPLICATIVE FACTORS, AND THE EXPONENTS OF THE
C   PURE TIME DELAYS.
C   READ (5,99) NFILE, M, XK, TAU1, (B(MM), MM=1,5), (T(MM), MM=1,5)
C   99 FORMAT(12, I1, 12(F5.2))
C
C   I=1
C   771 WRITE(6,1) NFILE, M, XK, TAU1, (B(MM), MM=1,5), (T(MM), MM=1,5)
C   1 FORMAT('1', ///, 3X,
C   2'COMPUTER RUN NUMBER ', I2, 9X, 'SATCHE DATA GENERATION' /, 3X,
C   7'THE NUMBER OF DELAYS IS ', I2, /, 3X,
C   8'THE CONSTANTS ARE: ', /, 3X,
C   1'K= ', F8.3, EX, 'TAU1= ', F8.3, //, 3X,
C   3'B(1) TO B(M): ', 5(F8.3, 2X) /, 3X,
C   5'T(1) TO T(M): ', 5(F8.3, 2X))
C
C   772 WRITE(6,3)
C   3 FORMAT(//, 4X, 'OMEGA OF F1', 8X, 'FE F1', 10X,
C   1'IM F1', 10X, 'RE F2', 10X, 'IM F2', 9X, 'OMEGA OF F2')
C
C   THE NEXT TWO STATEMENTS ARE USED FOR A
C   DESIRED PRINT-FORMAT.
C   N1=1
C   N1MAX=26-(M*2)
C   773 W=-PI
C   XDPI=(PI/DPI)/2.
C   GENERATION OF F(S).
C   4 X1(I)=-TAU1*W**2
C   61 Y1(I)=W
C   GENERATION OF F(EXP(S)), THE CURVE WITH DELAYS.
C   XT=T(1)
C   XX2=0.

```

```

      YY2=0.
      DO 67 K=1,M
      TEMPX2=B(K)*XK*COS(XT*W)
      TEMPY2=B(K)*XK*SIN(XT*W)
      IF(K-M) 69,68,68
69  XT=XT+T(K+1)
68  X2(I)=- (XX2+TEMPX2)
      XX2=-X2(I)
      Y2(I)=YY2+TEMPY2
      YY2=Y2(I)
67  CONTINUE
      WP=256.*W/PI
      OMEGA(I)=WP
777  WRITE(6,2) WP,X1(I),Y1(I),X2(I),Y2(I),WP
      2  FORMAT(3X,F6.1,'/256 PI',5X,4(1PE10.3,5X),3PF6.1,'/256 PI')
      IF(N1-N1MAX)8,10,10
      10  N1=0
      WRITE(6,1) NFILE,M,XK,TAU1,(B(MM),MM=1,5),(T(MM),MM=1,5)
      WRITE(6,3)
      8  N1=N1+1
88  IF((W+XDPI)-PI)12,9,9
      12  I=I+1
      7  W=W+PI/DPI
      GO TO 4
      9  CONTINUE
      19  WRITE(6,1) NFILE,M,XK,TAU1,(B(MM),MM=1,5),(T(MM),MM=1,5)
      97  WRITE(6,23)
      23  FORMAT(/,3X,'FINDING ALL POINTS OF INTERSECTION')
      96  WRITE(6,20)
      20  FORMAT(/,3X,'OBSERVE THE FOLLOWING CODES BELOW',/3X,
      1'N=0 : NOT A REGION OF INTERSECTION',/3X,
      2'N=1 : A REGION OF INTERSECTION')
      95  WRITE(6,21)
      21  FORMAT(/,4X, 'OMEGA OF F1',6X,'RE F1',8X,
      1'IM F1', 8X,'RE F2',8X,'IM F2', 7X,'OMEGA OF F2',4X,'N')
      98  IMAX=I
      JMAX=I
      IIMAX=0
      IIMIN=999
      N2=1
      N2MAX=19-(M*2)
      DO 32 I=1,IMAX
      CALL SCAN(I,X1(I),Y1(I),X2,Y2,N,J,JMAX)
      24  WRITE(6,26) OMEGA(I),X1(I),Y1(I),X2(J),Y2(J),OMEGA(J),N
      26  FORMAT(3X,F6.1,'/256 PI',3X,4(1PE10.3,3X),3PF6.1,'/256 PI',2X,I2)
      IF(N2-N2MAX)30,31,31
      31  N2=0
      WRITE(6,1) NFILE,M,XK,TAU1,(B(MM),MM=1,5),(T(MM),MM=1,5)
      WRITE(6,23)
      WRITE(6,20)
      WRITE(6,21)
      30  N2=N2+1
C  LOCATING THE OUTERMOST INTERSECTIONS.
      45  IF(N) 32,32,27
      27  IF(I-IIMIN)29,29,34
      29  IIMIN=I

```

```

      J1=J
34 IF(I-IIMAX) 32,33,33
33 IIMAX=I
      J2=J
32 CONTINUE
      WRITE(6,1) NFILE,M,XK,TAU1,(B(MM),MM=1,5),(T(MM),MM=1,5)
      WRITE(6,35)
35 FORMAT(/,3X,' THE OUTERMOST INTERSECTIONS OCCUR AT')
      WRITE(6,3)
      WRITE(6,2) OMEGA(IIMIN),X1(IIMIN),Y1(IIMIN),
1X2(J1),Y2(J1),CMEGA(J1)
      WRITE(6,2) OMEGA(IIMAX),X1(IIMAX),Y1(IIMAX),
1X2(J2),Y2(J2),CMEGA(J2)
C      CHECKING IF F2(W) IS OUTSIDE F1(W) FOR THE RANGE
C      OF OMEGA OF F2 VARYING FROM OMEGA SUB IIMIN TO
C      OMEGA SUB IIMAX
      WRITE(6,1) NFILE,M,XK,TAU1,(B(MM),MM=1,5),(T(MM),MM=1,5)
      WRITE(6,40)
40 FORMAT(/,3X,'**CHECKING IF F2 OF OMEGA SUB IIMIN THROUGH',/,3X,
1'F2 OF OMEGA SUB IIMAX LIES OUTSIDE OR INTERSECTS F1**')
      WRITE(6,20)
      WRITE(6,62)
62 FORMAT(3X,
1'L=-1 : X2 IS TO THE RIGHT OF F1,UNSTABLE CASE',/,3X,
2'L=1 : F2 INTERSECTS F1,UNSTABLE CASE',/,3X,
3'L=0 : F2 IS OUTSIDE OF F1,STABLE CASE')
      WRITE(6,55)
55 FORMAT(/,4X,' OMEGA OF F1',6X,' RE F1',8X,
1'IM F1',8X,' RE F2',8X,' IM F2',7X,' OMEGA OF F2',4X,' N',3X,' L')
      N3=1
      N3MAX=18-(M*2)
      DO 51 J=IIMIN,IIMAX
44 CALL SCAN(J,X2(J),Y2(J),X1,Y1,N,I,IIMAX)
      IF(N) 46,46,47
47 L=1
52 WRITE(6,53) OMEGA(I),X1(I),Y1(I),X2(J),Y2(J),
40MEGA(J),N,L
53 FORMAT(3X,F6.1,'/256 PI',3X,4(1PE10.3,3X),DPF6.1,'/256 PI',2X,I2,
12X,I2,' UNSTABLE')
      GO TO 56
46 IF(X2(J)-X1(I))71,71,72
72 L=-1
      WRITE(6,53) OMEGA(I),X1(I),Y1(I),X2(J),Y2(J),
40MEGA(J),N,L
      GO TO 56
71 L=0
49 WRITE(6,48) OMEGA(I),X1(I),Y1(I),X2(J),Y2(J),
40MEGA(J),N,L
48 FORMAT(3X,F6.1,'/256 PI',3X,4(1PE10.3,3X),DPF6.1,'/256 PI',2X,I2,
12X,I2,' STABLE')
56 IF(N3-N3MAX)54,64,64
64 N3=0
      WRITE(6,1) NFILE,M,XK,TAU1,(B(MM),MM=1,5),(T(MM),MM=1,5)
      WRITE(6,40)
      WRITE(6,20)
      WRITE(6,62)

```

```
      WRITE(6,55)
54  N3=N3+1
51  CONTINUE
C   PREPARING THE CALCOMP DIAGRAM
      IIIMIN=IIMIN
      ID=IIMAX-IIMIN
      DO 50 I=1, ID
        X1(I)=X1(IIMIN)
        Y1(I)=Y1(IIMIN)
        IIMIN=IIMIN+1
50  CONTINUE
      CALL SATCHE(X1,Y1,X2,Y2,ID      ,JMAX,BUFFER,4,XK,TAU1,NFILE,
18,Y,IIIMIN,IIMAX)
888 CONTINUE
      CALL PLOT(0.,0.,999)
999 STOP
      END
```

```
      SUBROUTINE SCAN(I,X1,Y1,X2,Y2,N,JX,JMAX)
      DIMENSION X2(1),Y2(1)
C     SUBROUTINE USED TO FIND POINTS OF INTERSECTION OF
C     TWO CURVES.
      DMIN=999.
      DO 3 J=1,JMAX
      D=SQRT((X1-X2(J))*#2+(Y1-Y2(J))*#2)
      IF(D-DMIN)2,2,3
2     DMIN=D
      JX=J
3     CONTINUE
      IF(DMIN-0.025)6,6,7
7     N=0
      RETURN
6     N=1
      RETURN
      END
```

```

SUBROUTINE SATCHE(X1,Y1,X2,Y2,J,I,BUFFER,M,XK,TAU1,NFILE,B,
1T,IIIMIN,IIIMAX)
DIMENSION X1(1),Y1(1),X2(1),Y2(1),BUFFER(1),B(1),T(1)
XL=9.
YL=7.
CALL PLOT (.5,.5,23)
CALL SCALE(X2,XL,I,1)
X1(J+1)=X2(I+1)
X1(J+2)=X2(I+2)
CALL SCALE(Y2,YL,I,1)
Y1(J+1)=Y2(I+1)
Y1(J+2)=Y2(I+2)
C DETERMINING ORIGIN OF AXES FOR X2,Y2
C (LOCATING HEIGHT OF X-AXIS)
XSUBX=0.
IF(Y2(I+1)-0.) 1,2,2
2 XSUBY=0.0
GO TO 5
1 D=Y2(I+1)+YL*Y2(I+2)
IF (D-0.) 3,3,4
4 XSUBY=-Y2(I+1)/Y2(I+2)
GO TO 5
3 XSUBY=YL
C
C LOCATING WIDTH OF Y-AXIS
5 YSUBX=0.
IF(X2(I+1)-0.) 6,7,7
7 YSUBX=0.
GO TO 10
6 D=X2(I+1)+XL*X2(I+2)
IF(D-0.) 8,8,11
11 YSUBX=-X2(I+1)/X2(I+2)
GO TO 10
8 YSUBX=XL
10 CALL AXIS(XSUBX,XSUBY,'
1XL,0.,X2(I+1),X2(I+2))
CALL AXIS(YSUBX,YSUBY,'
1Y2(I+1),Y2(I+2))
CALL LINE(X2,Y2,1,1,0,0)
CALL LINE(X1,Y1,J,1,+3,03)
IID=IIIMAX-IIIMIN
DO 12 JJ=1,IID
X2(JJ)=X2(IIIMIN)
Y2(JJ)=Y2(IIIMIN)
IIIMIN=IIIMIN+1
12 CONTINUE
X2(JJ+1)=X2(I+1)
X2(JJ+2)=X2(I+2)
Y2(JJ+1)=Y2(I+1)
Y2(JJ+2)=Y2(I+2)
CALL LINE(X2,Y2,JJ,1,-3,11)
CALL PLOT(YSUBX+0.20,YL,23)
CALL PLOT(4.8,0.,22)
CALL PLOT(0.,-1.1,22)
CALL PLOT(-4.8,0.,22)
REAL F1,F2',-32.
IMAG F1,F2',25,YL,90..

```

```

CALL PLOT(0.,1.1,22)
CALL SYMBOL(0.15,-0.25,0.15,22HS,ATCHE DIAGRAM ,0.,22)
CALL NUMBER(1.6,-0.4,0.1,XK,0.,2)
CALL NUMBER(2.7,-0.4,0.1,TAU1,0.,2)
CALL SYMBOL(2.8,-0.25,0.10,17H-DELAY(S) PROBLEM,0.,17)
CALL NUMBER(2.700,-0.25,0.10,FLOAT(M),0.,-1)
CALL SYMBOL(0.15,-0.4,0.1,30HCONSTANTS: K= ,TAU1= ,0.,30)
CALL SYMBOL(0.15,-0.55,0.10,43HTHE EXPONENTS FOR THE PURE TIME DEL
1AYS ARE:,0.,43)
CALL SYMBOL(0.15,-0.85,0.1,45HTHE FEEDBACK COEFFICIENTS FOR THE DE
2LAYS ARE:,0.,45)
XN=0.
DO 14 NN=1,5
CALL NUMBER(0.15+XN,-0.7,0.1,T(NN),0.,1)
XN=XN+0.8
14 CONTINUE
XN=0.
DO 15 NN=1,5
CALL NUMBER(0.15+XN,-1.,0.1,8(NN),0.,1)
XN=XN+0.8
15 CONTINUE
CALL SYMBOL(1.2,-6.7,0.1,27HCALC CMP COMPUTER-DRAWN PLOT,0.,27)
CALL SYMBOL(1.2,-6.85,0.1,37HPR OGRAM-PLQT DESIGN : JACQUES PRES
1S,0.,37)
CALL SYMBOL(1.2,-7.00,0.1,12HDATE: 8/1/72,0.,12)
CALL PLOT(-4.0,-0.2,3)
CALL PLOT(-3.70,-0.2,2)
CALL SYMBOL(-3.65, -0.25,0.10,27HF(EXP(S)):CURVE WITH DELAYS,
10.,27)
CALL SYMBOL(-4.,-0.65,0.1,30H++++ F(S):CURVE WITH NO DELAYS,0.,30)
CALL SYMBOL(-4.,-1.05,0.1,37H*** PART OF F(EXP(S)) CORRESPONDING
1,0.,37)
CALL SYMBOL(-3.65,-1.20,0.1,28HTC OMEGA OF F(S) BOUNDED BY ,0.,28)
CALL SYMBOL(-3.65,-1.35,0.1,28HITS OUTER INTERSECTIONS WITH,0.,28)
CALL SYMBOL(-3.65,-1.50,0.1,9HF(EXP(S)),0.,9)
CALL PLOT(10.,-7.5,-23)
RETURN
END

```


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APPENDIX F

THE MATHEMATICAL FUNCTIONS USED IN THE CSMP SIMULATION
(SEE FIG. 17)

<u>Computer Notation</u>	<u>Mathematical Notation</u>
A. Y = STEP (P) (STEP FUNCTION)	$Y = 0 \quad t < P$ $Y = 1 \quad t \geq P$
B. Y = REALPL (IC, P, X) Y(0) = IC (SIMPLE LAG)	$PY = \dot{Y} = X$ $\left(\frac{1}{P s + 1} \right)$
C. Y = INTGRL (IC, X) Y (0) = IC (INTEGRAL)	$Y = \int_0^t X dt + IC$ $\left(\frac{1}{s} \right)$
D. Y = DELAY (N, P, X) P = DELAY TIME N = NUMBER OF POINTS SAMPLED IN INTERVAL P (INTEGER CONSTANT) DEAD TIME (DELAY)	$Y(t) = X(t - P) \quad t \geq P$ $Y = 0 \quad t < P$ (e^{-sP})

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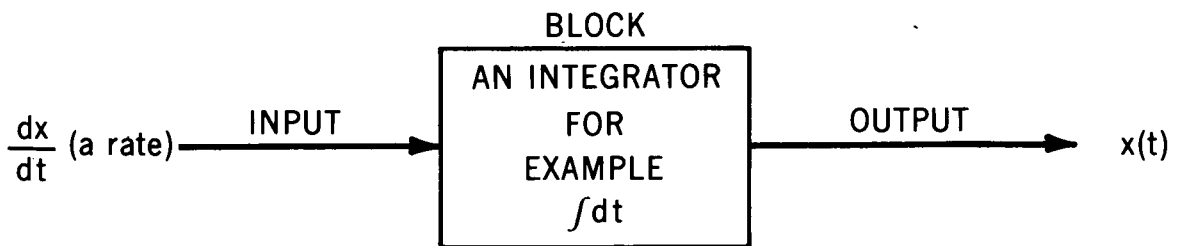


Figure 1a. The mathematical box as a component of a system.

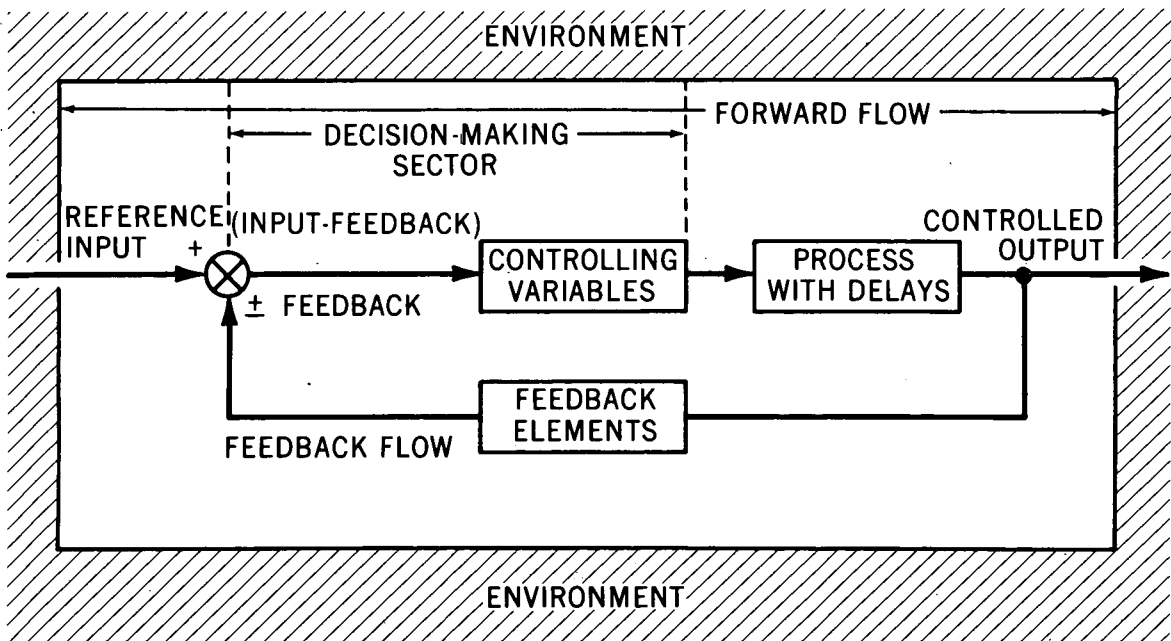


Figure 1b. A generalized feedback control system.

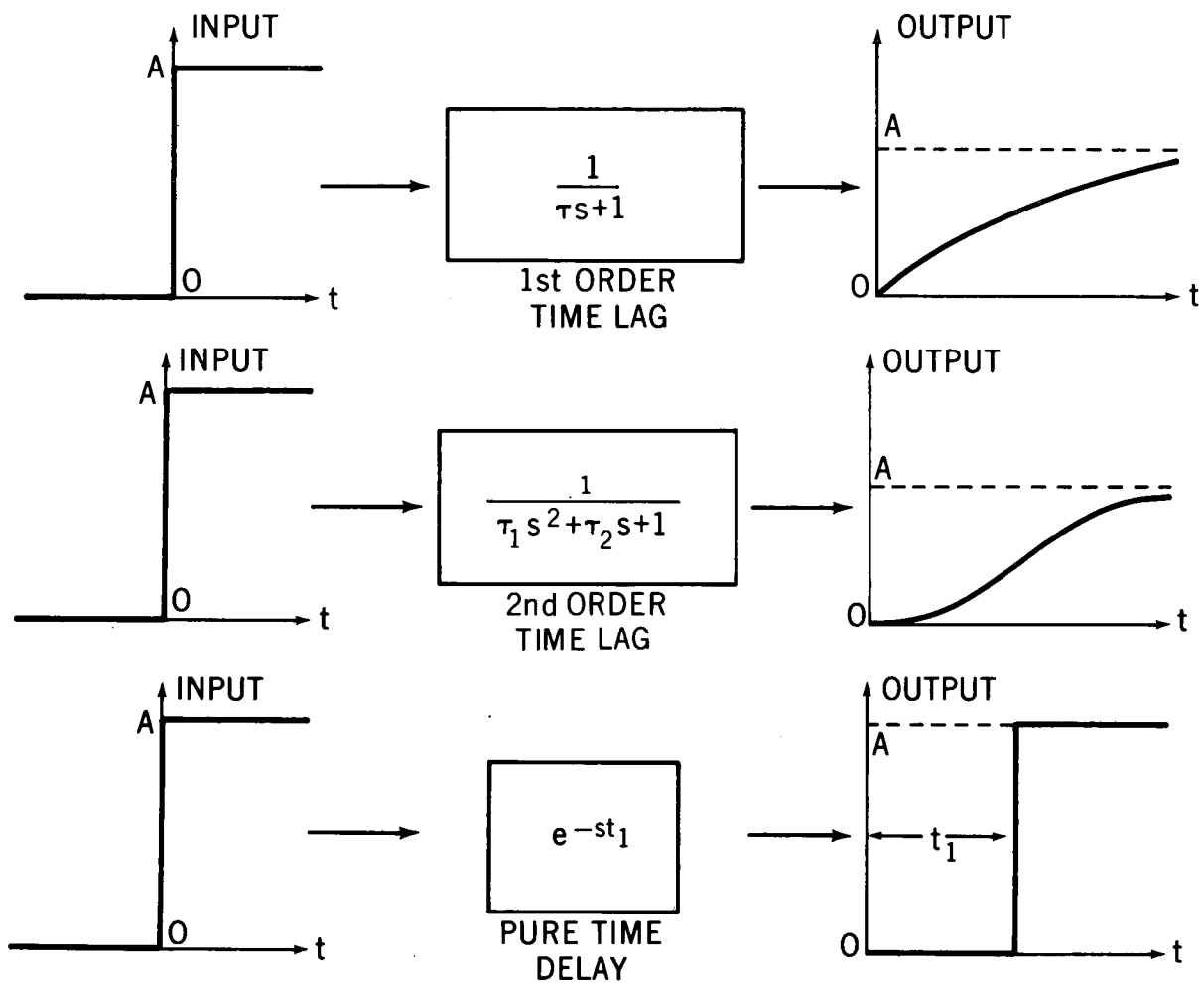


Figure 2. Various time lag and pure time delay transformation functions.

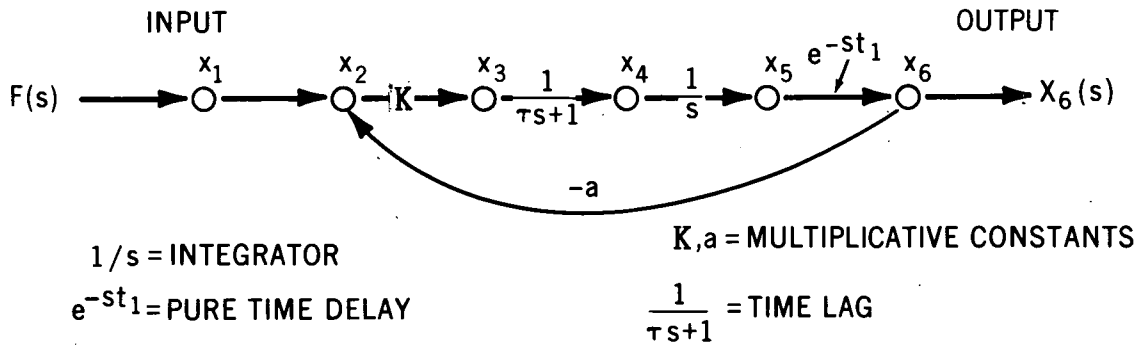


Figure 3a. The network graph for a problem with one pure time delay.

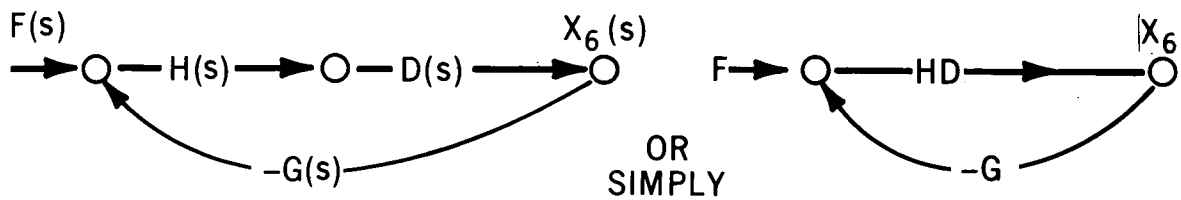


Figure 3b. The algebraic diagram for Figure 3a.

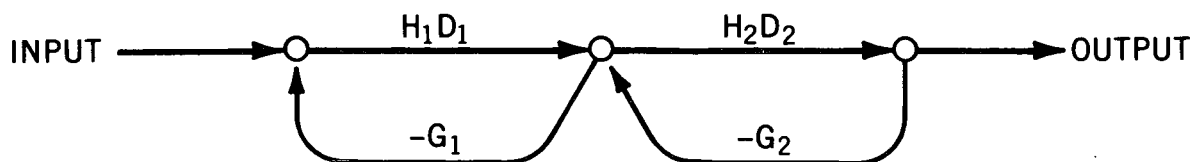


Figure 4a. A trivial extension to two pure time delays.

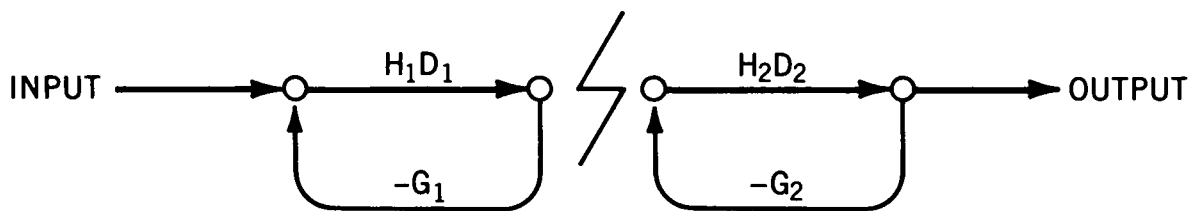


Figure 4b. The reduction of Figure 4a to two single delays.

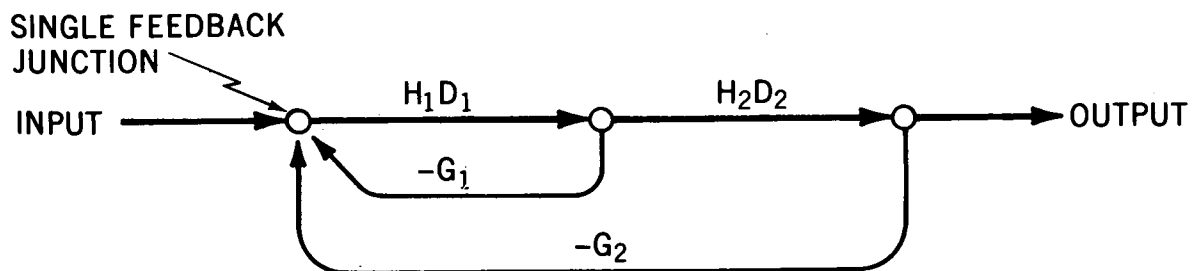


Figure 4c. The non-trivial extension to two pure time delays.

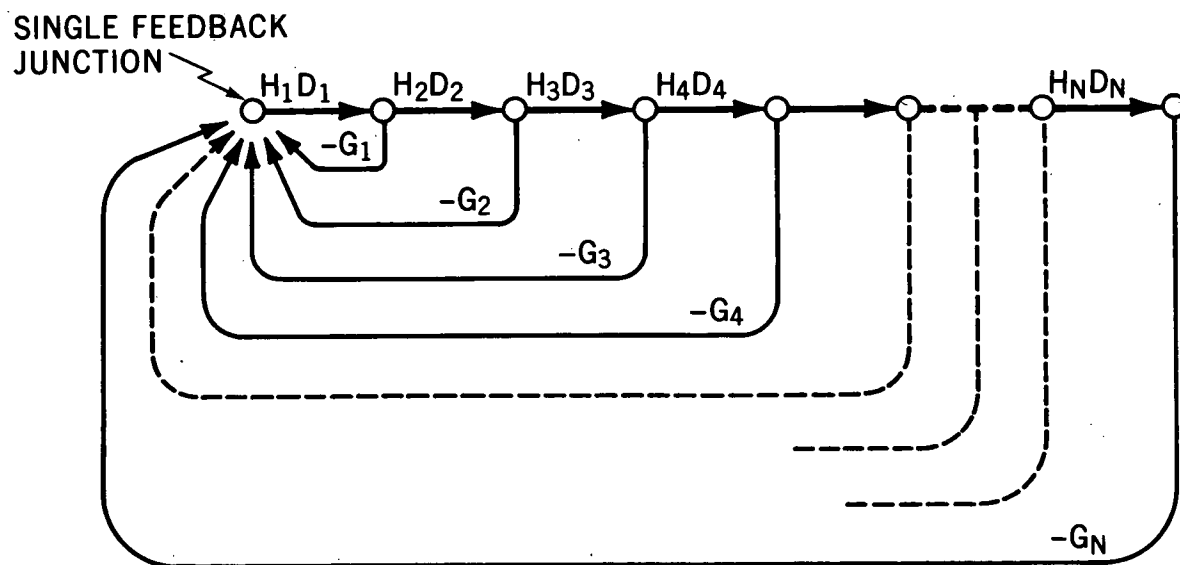


Figure 4d. The general non-trivial extension to N -pure time delays.

PADE TABLE FOR e^{-x}

$\frac{1}{1}$	$\frac{1-x}{1}$	$\frac{1-x+\frac{x^2}{2!}}{1}$	$\frac{1-x+\frac{x^2}{2!}-\frac{x^3}{3!}}{1}$
$\frac{1}{1+x}$	$\frac{1-\frac{1}{2}x}{1+\frac{1}{2}x}$	$\frac{1-\frac{2}{3}x+\frac{1}{3}\frac{x^2}{2!}}{1+\frac{1}{3}x}$	$\frac{1-\frac{3}{4}x+\frac{2}{4}\frac{x^2}{2!}-\frac{1}{4}\frac{x^3}{3!}}{1+\frac{1}{4}x}$
$\frac{1}{1+x+\frac{x^2}{2!}}$	$\frac{1-\frac{1}{3}x}{1+\frac{2}{3}x+\frac{1}{3}\frac{x^2}{2!}}$	$\frac{1-\frac{1}{2}x+\frac{1}{6}\frac{x^2}{2!}}{1+\frac{1}{2}x+\frac{1}{6}\frac{x^2}{2!}}$	$\frac{1-\frac{3}{5}x+\frac{3}{10}\frac{x^2}{2!}-\frac{1}{10}\frac{x^3}{3!}}{1+\frac{2}{5}x+\frac{1}{10}\frac{x^2}{2!}}$
$\frac{1}{1+x+\frac{x^2}{2!}+\frac{x^3}{3!}}$	$\frac{1-\frac{1}{4}x}{1+\frac{3}{4}x+\frac{2}{4}\frac{x^2}{2!}+\frac{1}{4}\frac{x^3}{3!}}$	$\frac{1-\frac{2}{5}x+\frac{1}{10}\frac{x^2}{2!}}{1+\frac{3}{5}x+\frac{3}{10}\frac{x^2}{2!}+\frac{1}{10}\frac{x^3}{3!}}$	$\frac{1-\frac{1}{2}x+\frac{1}{5}\frac{x^2}{2!}-\frac{1}{20}\frac{x^3}{3!}}{1+\frac{1}{2}x+\frac{1}{5}\frac{x^2}{2!}+\frac{1}{20}\frac{x^3}{3!}}$
$\frac{1}{1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}}$	$\frac{1-\frac{1}{5}x}{1+\frac{4}{5}x+\frac{3}{5}\frac{x^2}{2!}+\frac{2}{5}\frac{x^3}{3!}+\frac{1}{5}\frac{x^4}{4!}}$	$\frac{1-\frac{1}{3}x+\frac{1}{15}\frac{x^2}{2!}}{1+\frac{2}{3}x+\frac{2}{5}\frac{x^2}{2!}+\frac{1}{5}\frac{x^3}{3!}+\frac{1}{15}\frac{x^4}{4!}}$	$\frac{1-\frac{3}{7}x+\frac{1}{7}\frac{x^2}{2!}-\frac{1}{35}\frac{x^3}{3!}}{1+\frac{4}{7}x+\frac{2}{7}\frac{x^2}{2!}+\frac{4}{35}\frac{x^3}{3!}+\frac{1}{35}\frac{x^4}{4!}}$

Figure 5. The Padé table. ($x = s$)

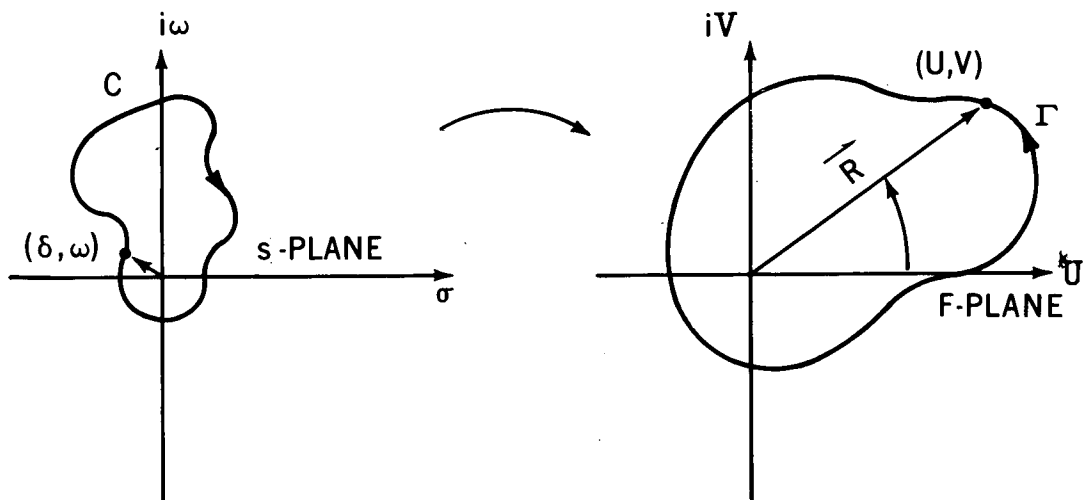


Figure 6a. The conformal mapping from the s -plane onto the $F(s, e^s)$ -plane.

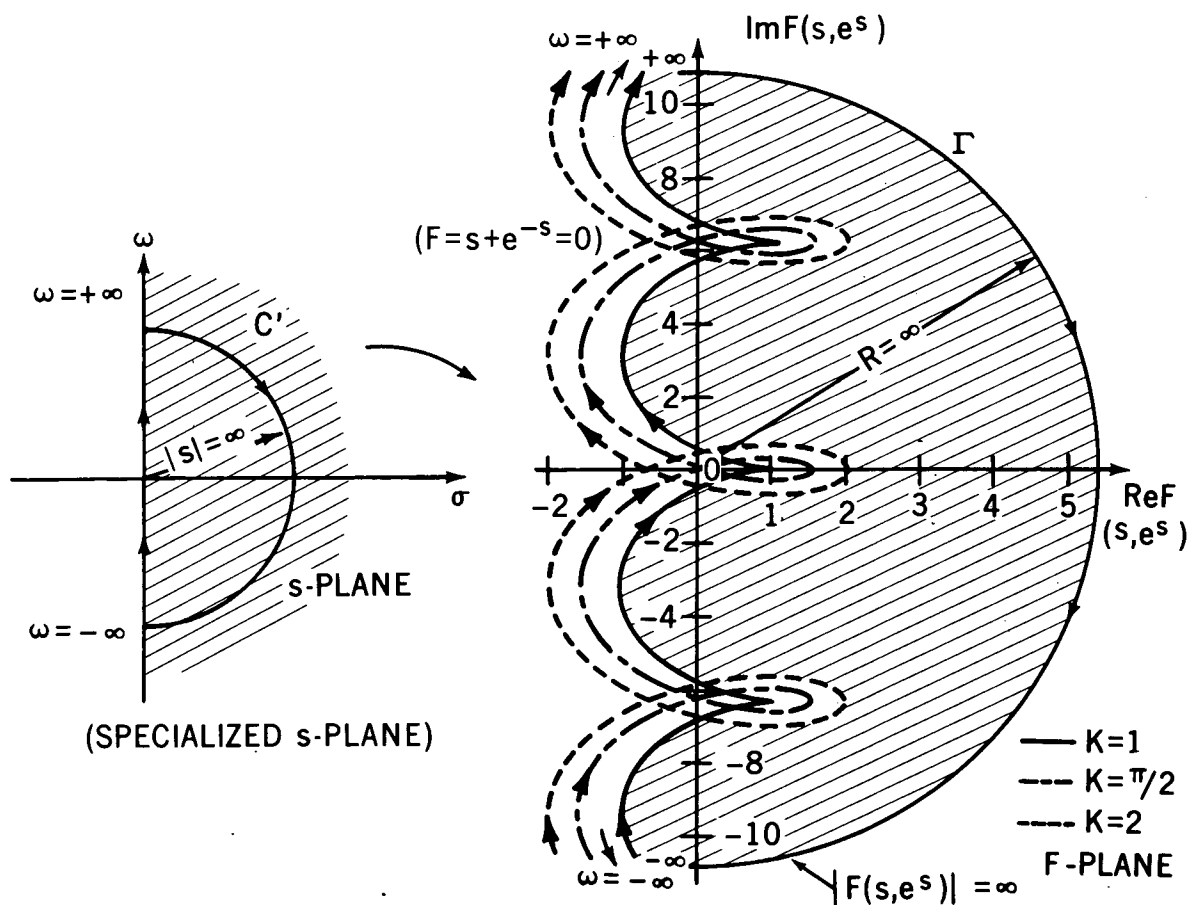


Figure 6b. The Nyquist plot for $F(s, e^s) = s + e^{-s} = 0$.

(A,B are moving points)

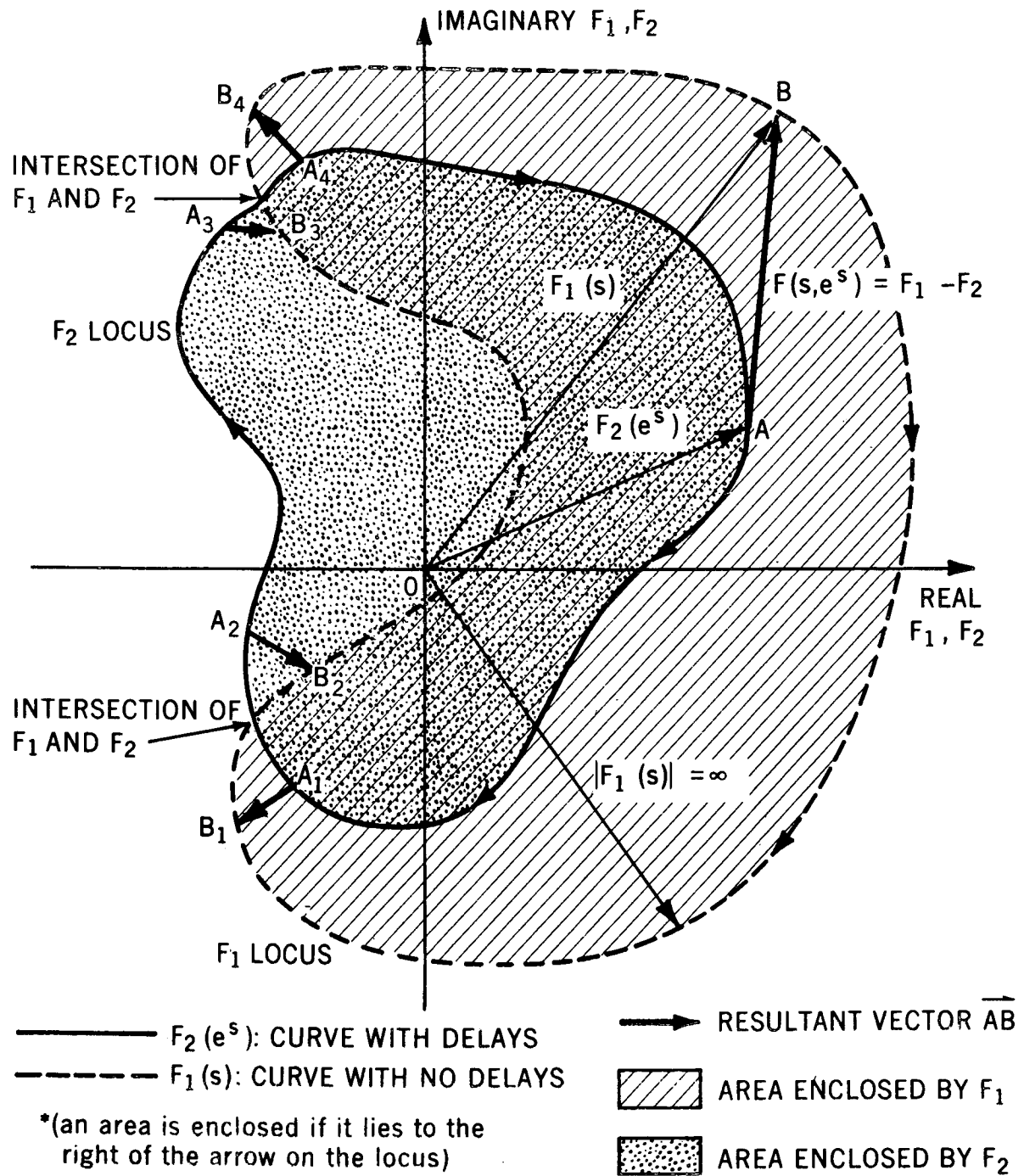


Figure 6c. The Satche diagram.

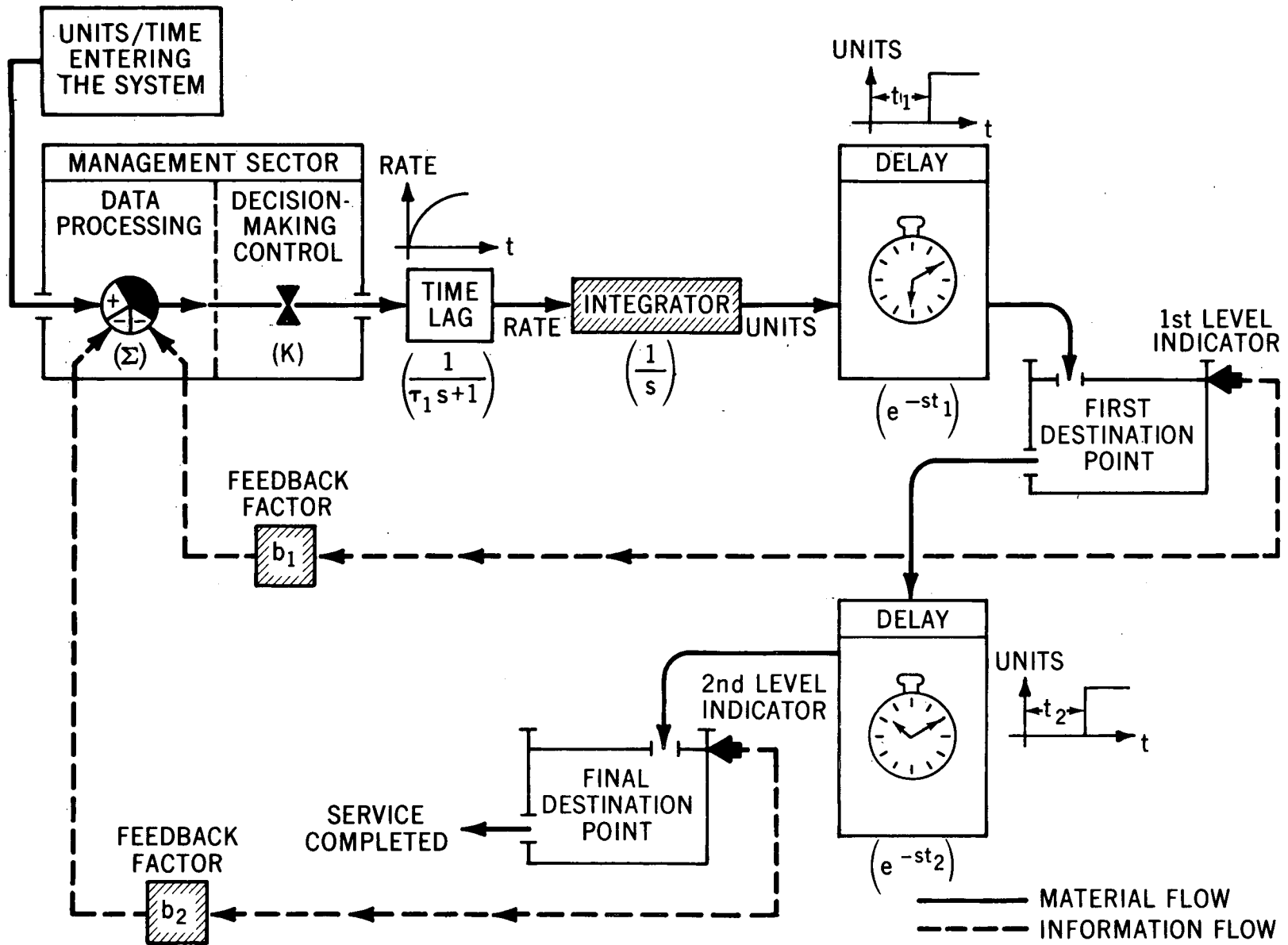


Figure 7. A proposed model for a network flow hindered by two pure time delays.

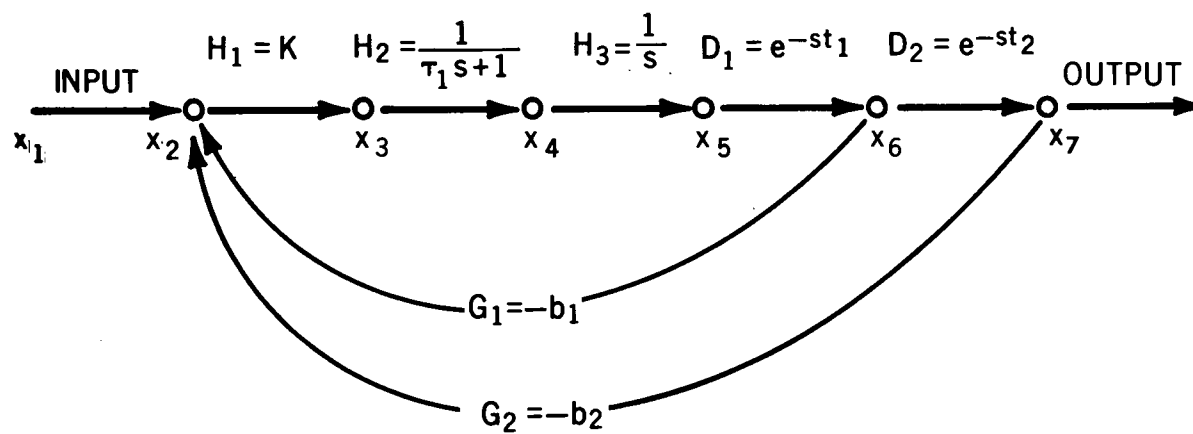
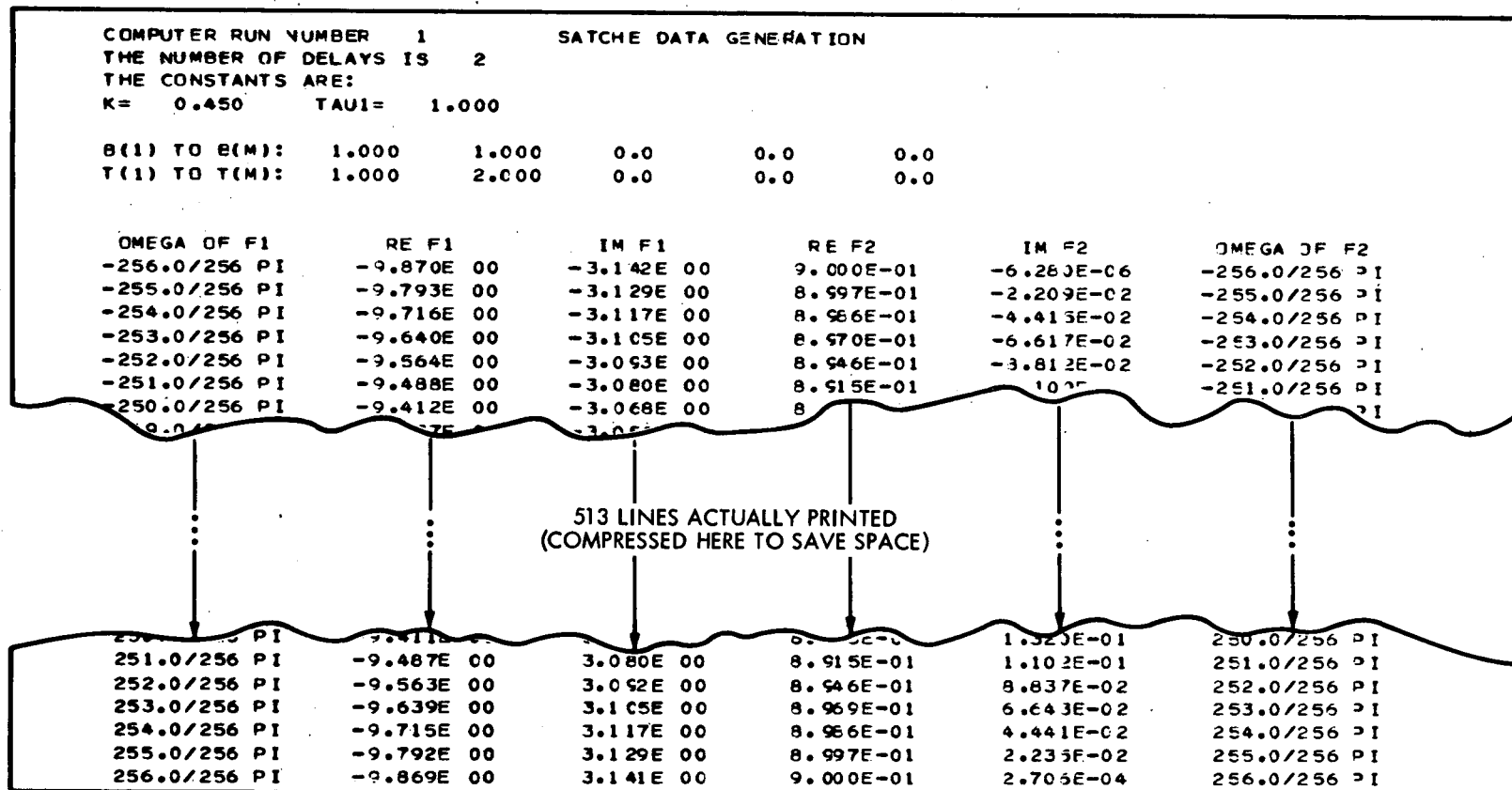


Figure 8. The network graph for the model in Figure 7.



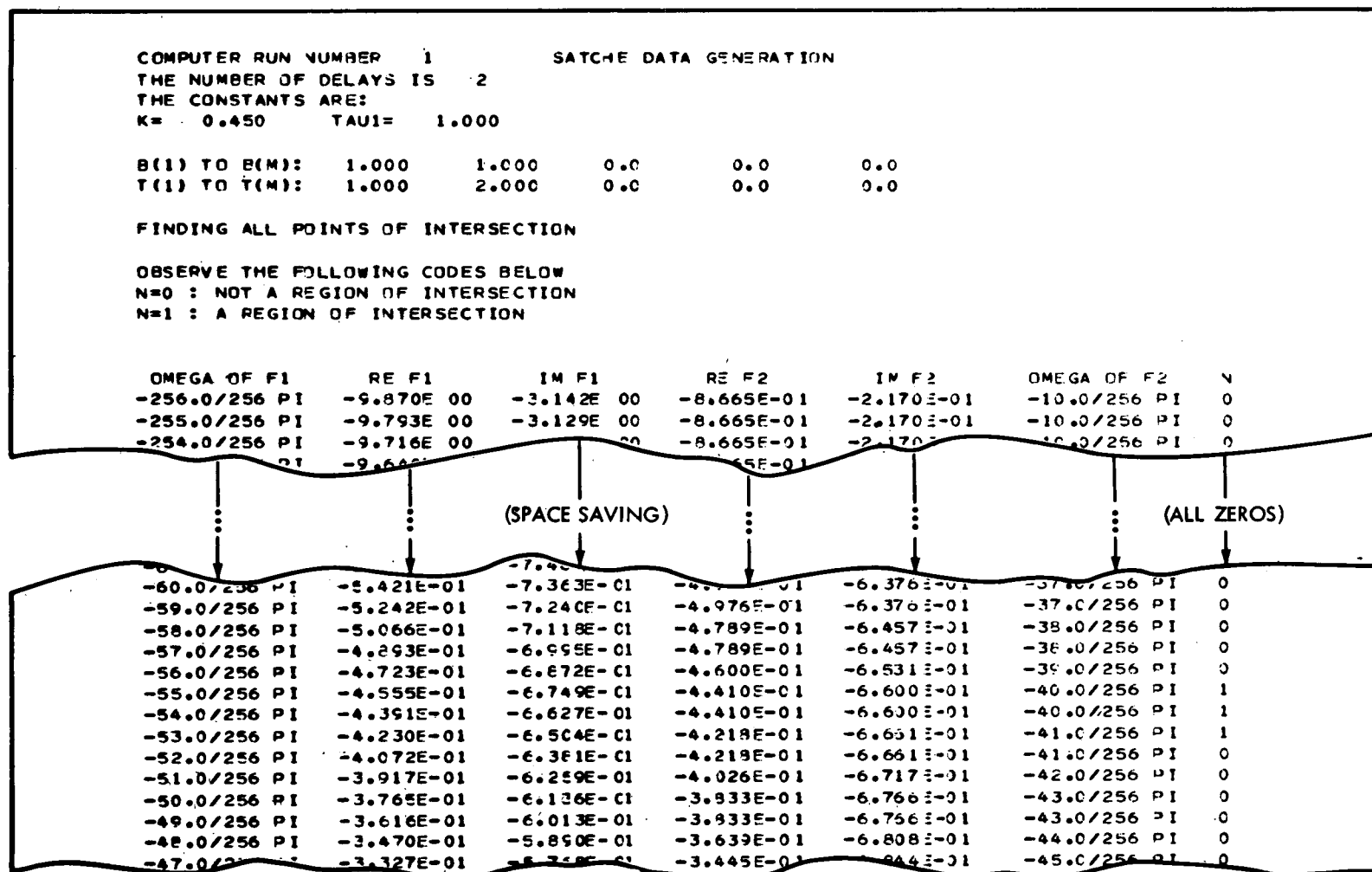


Figure 10. Computer output displaying where F_1 and F_2 intersect numerically.

OMEGA OF F1	RE F1	IM F1	RE F2	IM F2	OMEGA OF F2	N
-46.0/256 PI	-3.197E-01	-5.645E-C1	-3.845E-01	-6.844E-01	-45.0/256 PI	0
-45.0/256 PI	-3.049E-01	-5.522E-C1	-3.251E-01	-6.974E-01	-46.0/256 PI	0
-44.0/256 PI	-2.915E-01	-5.400E-C1	-1.932E-01	-4.750E-01	-177.0/256 PI	0
-43.0/256 PI	-2.784E-01	-5.277E-C1	-1.932E-01	-4.750E-01	-177.0/256 PI	0
-42.0/256 PI	-2.656E-01	-5.154E-C1	-1.914E-01	-4.619E-01	-176.0/256 PI	0
-41.0/256 PI	-2.531E-01	-5.031E-C1	-1.914E-01	-4.619E-01	-176.0/256 PI	0
-40.0/256 PI	-2.409E-01	-4.909E-C1	-1.914E-01	-4.619E-01	-176.0/256 PI	0
-39.0/256 PI	-2.290E-01	-4.786E-C1	-1.914E-01	-4.619E-01	-176.0/256 PI	0
-38.0/256 PI	-2.175E-01	-4.663E-C1	-1.989E-01	-4.437E-01	-175.0/256 PI	0
-37.0/256 PI	-2.062E-01	-4.540E-C1	-1.989E-01	-4.487E-01	-175.0/256 PI	1
-36.0/256 PI	-1.952E-01	-4.418E-C1	-1.989E-01	-4.437E-01	-175.0/256 PI	1
-35.0/256 PI	-1.845E-01	-4.295E-01	-2.059E-01	-4.353E-01	-174.0/256 PI	1
-34.0/256 PI	-1.741E-01	-4.172E-C1	-2.059E-01	-4.353E-01	-174.0/256 PI	0
-33.0/256 PI	-1.640E-01	-4.050E-C1	-2.123E-01	-4.217E-01	-173.0/256 PI	0
(SPACE SAVING)						
-4.0/256 PI	-2.409E-03	-6.135E-02	7.149E-06	0.0	-128.0/256 PI	0
-3.0/256 PI	-1.354E-03	-4.908E-02	7.149E-06	0.0	-128.0/256 PI	0
-2.0/256 PI	-6.018E-04	-2.453E-C2	7.149E-06	0.0	-128.0/256 PI	1
-1.0/256 PI	-1.503E-04	-1.226E-02	7.149E-06	0.0	-128.0/256 PI	1
0.0/256 PI	-1.451E-10	1.205E-05	7.149E-06	0.0	-128.0/256 PI	1
1.0/256 PI	-1.509E-04	1.228E-02	7.149E-06	0.0	-128.0/256 PI	1
2.0/256 PI	-6.030E-04	2.456E-C2	7.149E-06	0.0	-128.0/256 PI	1
3.0/256 PI	-1.356E-03	3.663E-C2	7.149E-06	0.0	-128.0/256 PI	0
4.0/256 PI	-2.411E-03	4.910E-C2	7.149E-06	0.0	-128.0/256 PI	0
5.0/256 PI	-3.766E-03	6.137E-C2	7.149E-06	0.0	-128.0/256 PI	0
6.0/256 PI	-5.423E-03	7.364E-C2	-1.101E-02	2.695E-04	129.0/256 PI	0
7.0/256 PI	-7.381E-03	8.591E-C2	-1.101E-02	2.695E-04	129.0/256 PI	0
8.0/256 PI	-9.641E-03	9.815E-C2	-2.203E-02	1.031E-03	130.0/256 PI	0
9.0/256 PI	-1.220E-02	1.105E-C1	-2.203E-02	1.031E-03	130.0/256 PI	0
10.0/256 PI	-1.506E-02	1.227E-01	-3.301E-02	2.432E-03	131.0/256 PI	0
11.0/256 PI	-1.823E-02	1.350E-C1	-4.392E-02	4.322E-03	132.0/256 PI	0
12.0/256 PI	-2.165E-02	1.473E-C1	-5.474E-02	6.747E-03	133.0/256 PI	0

Figure 10. (Continued)

OMEGA OF F1	RE F1	IM F1	RE F2	IM F2	OMEGA OF F2	N
29.0/256 PI	-1.267E-01	3.559E-01	-2.234E-01	3.942E-01	171.0/256 PI	0
30.0/256 PI	-1.355E-01	3.682E-01	-2.234E-01	3.942E-01	171.0/256 PI	0
31.0/256 PI	-1.447E-01	3.804E-01	-2.181E-01	4.030E-01	172.0/256 PI	0
32.0/256 PI	-1.542E-01	3.927E-01	-2.123E-01	4.216E-01	173.0/256 PI	0
33.0/256 PI	-1.640E-01	4.050E-01	-2.123E-01	4.216E-01	173.0/256 PI	0
34.0/256 PI	-1.741E-01	4.173E-01	-2.059E-01	4.352E-01	174.0/256 PI	0
35.0/256 PI	-1.845E-01	4.295E-01	-2.059E-01	4.352E-01	174.0/256 PI	1
36.0/256 PI	-1.952E-01	4.418E-01	-1.989E-01	4.486E-01	175.0/256 PI	1
37.0/256 PI	-2.062E-01	4.541E-01	-1.989E-01	4.486E-01	175.0/256 PI	1
38.0/256 PI	-2.175E-01	4.663E-01	-1.989E-01	4.486E-01	175.0/256 PI	0
39.0/256 PI	-2.291E-01	4.786E-01	-1.914E-01	4.619E-01	176.0/256 PI	0
40.0/256 PI	-2.410E-01	4.909E-01	-1.914E-01	4.619E-01	176.0/256 PI	0
41.0/256 PI	-2.532E-01		-1.914E-01			
...	(SPACE SAVING)	(ALL ZEROS)
50.0/256 PI		6.100E-01		6.100E-01	42.0/256 PI	
51.0/256 PI	-3.917E-01	6.259E-01	-4.026E-01	6.661E-01	41.0/256 PI	0
52.0/256 PI	-4.072E-01	6.381E-01	-4.218E-01	6.661E-01	41.0/256 PI	1
53.0/256 PI	-4.230E-01	6.504E-01	-4.218E-01	6.661E-01	41.0/256 PI	1
54.0/256 PI	-4.392E-01	6.627E-01	-4.410E-01	6.600E-01	40.0/256 PI	1
55.0/256 PI	-4.556E-01	6.750E-01	-4.410E-01	6.600E-01	40.0/256 PI	1
56.0/256 PI	-4.723E-01	6.872E-01	-4.600E-01	6.532E-01	39.0/256 PI	
57.0/256 PI	-4.893E-01		-4.780E-01			
...	(SPACE SAVING)	(ALL ZEROS)
254.0/256 PI	-9.710E-00	3.111E-00	-8.665E-01	2.171E-01	10.0/256 PI	0
255.0/256 PI	-9.792E-00	3.129E-00	-8.665E-01	2.171E-01	10.0/256 PI	0
256.0/256 PI	-9.869E-00	3.141E-00	-8.665E-01	2.171E-01	10.0/256 PI	0

Figure 10. (Concluded).

COMPUTER RUN NUMBER 1 SATCHE DATA GENERATION
 THE NUMBER OF DELAYS IS 2
 THE CONSTANTS ARE:
 K= 0.450 TAU1= 1.000

B(1) TO B(M):	1.000	1.000	0.0	0.0	0.0
T(1) TO T(M):	1.000	2.000	0.0	0.0	0.0

8

THE OUTERMOST INTERSECTIONS OCCUR AT

OMEGA OF F1	RE F1	IM F1	RE F2	IM F2	OMEGA OF F2
-55.0/256 PI	-4.555E-01	-6.749E-01	-4.410E-01	-6.600E-01	-40.0/256 PI
55.0/256 PI	-4.556E-01	6.750E-01	-4.410E-01	6.600E-01	40.0/256 PI

Figure 11. Computer output displaying the numerical values at the outermost intersections.

COMPUTER RUN NUMBER 2 SATCHE DATA GENERATION
 THE NUMBER OF DELAYS IS 2
 THE CONSTANTS ARE:
 K= 0.450 TAU1= 1.000

B(1) TO B(M): 1.000 1.000 0.0 0.0 0.0
 T(1) TO T(M): 1.000 2.000 0.0 0.0 0.0

**CHECKING IF F2 OF OMEGA SUB IIMIN THROUGH
 F2 OF OMEGA SUB IIMAX LIES OUTSIDE OR INTERSECTS F1**

OBSERVE THE FOLLOWING CODES BELOW
 N=0 : NOT A REGION OF INTERSECTION
 N=1 : A REGION OF INTERSECTION
 L=-1 : X2 IS TO THE RIGHT OF F1, UNSTABLE CASE
 L=1 : F2 INTERSECTS F1, UNSTABLE CASE
 L=0 : F2 IS OUTSIDE OF F1, STABLE CASE

OMEGA OF F1	RE F1	IM F1	RE F2	IM F2	OMEGA OF F2	N	L
-44.0/256 PI	-2.915E-01	-5.400E-01	-1.540E-01	-6.856E-01	-55.0/256 PI	0	-1 UNSTABLE
-45.0/256 PI	-3.049E-01	-5.522E-01	-1.724E-01	-6.883E-01	-54.0/256 PI	0	-1 UNSTABLE
-45.0/256 PI	-3.049E-01	-5.522E-01	-1.910E-01	-6.903E-01	-53.0/256 PI	0	-1 UNSTABLE
-46.0/256 PI	-3.187E-01	-5.645E-01	-2.099E-01	-6.918E-01	-52.0/256 PI	0	-1 UNSTABLE
-47.0/256 PI	-3.327E-01	-5.768E-01	-2.288E-01	-6.926E-01	-51.0/256 PI	0	-1 UNSTABLE
-48.0/256 PI	-3.470E-01	-5.890E-01	-2.479E-01	-6.928E-01	-50.0/256 PI	0	-1 UNSTABLE
-48.0/256 PI	-3.470E-01	-5.890E-01	-2.671E-01	-6.924E-01	-49.0/256 PI	0	-1 UNSTABLE
-49.0/256 PI	-3.616E-01	-6.013E-01	-2.864E-01	-6.914E-01	-48.0/256 PI	0	-1 UNSTABLE
-50.0/256 PI	-3.765E-01	-6.136E-01	-3.057E-01	-6.897E-01	-47.0/256 PI	0	-1 UNSTABLE
-50.0/256 PI	-3.765E-01	-6.136E-01	-3.251E-01	-6.874E-01	-46.0/256 PI	0	-1 UNSTABLE
-51.0/256 PI	-3.917E-01	-6.259E-01	-3.445E-01	-6.844E-01	-45.0/256 PI	0	-1 UNSTABLE
-52.0/256 PI	-4.072E-01	-6.381E-01	-3.639E-01	-6.808E-01	-44.0/256 PI	0	-1 UNSTABLE
-52.0/256 PI	-4.072E-01	-6.381E-01	-3.833E-01	-6.756E-01	-43.0/256 PI	0	-1 UNSTABLE
-53.0/256 PI	-4.230E-01	-6.504E-01	-4.026E-01	-6.717E-01	-42.0/256 PI	0	-1 UNSTABLE

Figure 12. Computer output displaying regions of stability and instability.

COMPUTER RUN NUMBER 2 SATCHE DATA GENERATION
 THE NUMBER OF DELAYS IS 2
 THE CONSTANTS ARE:
 K= 0.450 TAU1= 1.000

B(1) TO B(M): 1.000 1.000 0.0 0.0 0.0
 T(1) TO T(M): 1.000 2.000 0.0 0.0 0.0

**CHECKING IF F2 OF OMEGA SUB IIMIN THROUGH
 F2 OF OMEGA SUB IIMAX LIES OUTSIDE OR INTERSECTS F1**

OBSERVE THE FOLLOWING CODES BELOW
 N=0 : NOT A REGION OF INTERSECTION
 N=1 : A REGION OF INTERSECTION
 L=-1 : X2 IS TO THE RIGHT OF F1, UNSTABLE CASE
 L=1 : F2 INTERSECTS F1, UNSTABLE CASE
 L=0 : F2 IS OUTSIDE OF F1, STABLE CASE

OMEGA OF F1	RE F1	IM F1	RE F2	IM F2	OMEGA OF F2	N	L
-53.0/256 PI	-4.230E-01	-6.504E-01	-4.218E-01	-6.661E-01	-41.0/256 PI	1	1 UNSTABLE
-54.0/256 PI	-4.391E-01	-6.627E-01	-4.410E-01	-6.600E-01	-40.0/256 PI	1	1 UNSTABLE
-55.0/256 PI	-4.555E-01	-6.749E-01	-4.600E-01	-6.531E-01	-39.0/256 PI	1	1 UNSTABLE
-55.0/256 PI	-4.555E-01	-6.749E-01	-4.789E-01	-6.457E-01	-38.0/256 PI	0	0 STABLE
-56.0/256 PI	-4.723E-01	-6.872E-01	-4.976E-01	-6.376E-01	-37.0/256 PI	0	0 STABLE
-56.0/256 PI	-4.723E-01	-6.872E-01	-5.162E-01	-6.289E-01	-36.0/256 PI	0	0 STABLE
-57.0/256 PI	-4.893E-01	-6.995E-01	-5.345E-01	-6.196E-01	-35.0/256 PI	0	0 STABLE
-57.0/256 PI	-4.893E-01	-6.995E-01	-5.526E-01	-6.096E-01	-34.0/256 PI	0	0 STABLE
-57.0/256 PI	-4.893E-01	-6.995E-01	-5.704E-01	-5.991E-01	-33.0/256 PI	0	0 STABLE
-58.0/256 PI	-5.066E-01	-7.118E-01	-5.880E-01	-5.879E-01	-32.0/256 PI	0	0 STABLE
-58.0/256 PI	-5.066E-01	-7.118E-01	-6.052E-01	-5.762E-01	-31.0/256 PI	0	0 STABLE
-58.0/256 PI	-5.066E-01	-7.118E-01	-6.222E-01	-5.639E-01	-30.0/256 PI	0	0 STABLE
-59.0/256 PI	-5.242E-01	-7.240E-01	-6.388E-01	-5.510E-01	-29.0/256 PI	0	0 STABLE
-59.0/256 PI	-5.242E-01	-7.240E-01	-6.551E-01	-5.376E-01	-28.0/256 PI	0	0 STABLE

Figure 12. (Continued).

COMPUTER RUN NUMBER 2 SATCHEL DATA GENERATION
 THE NUMBER OF DELAYS IS 2
 THE CONSTANTS ARE:
 K= 0.450 TAU1= 1.000

B(1) TO B(M): 1.000 1.000 0.0 0.0 0.0
 T(1) TO T(M): 1.000 2.000 0.0 0.0 0.0

**CHECKING IF F2 OF OMEGA SUB IIMIN THROUGH
 F2 OF OMEGA SUB IIMAX LIES OUTSIDE OR INTERSECTS F1**

OBSERVE THE FOLLOWING CODES BELOW
 N=0 : NOT A REGION OF INTERSECTION
 N=1 : A REGION OF INTERSECTION
 L=-1 : X2 IS TO THE RIGHT OF F1, UNSTABLE CASE
 L=1 : F2 INTERSECTS F1, UNSTABLE CASE
 L=0 : F2 IS OUTSIDE OF F1, STABLE CASE

OMEGA OF F1	RE F1	IM F1	RE F2	IM F2	OMEGA OF F2	N	L
-59.0/256 PI	-5.242E-01	-7.240E-01	-6.709E-01	-5.236E-01	-27.0/256 PI	0	0 STABLE
-60.0/256 PI	-5.421E-01	-7.363E-01	-6.864E-01	-5.091E-01	-26.0/256 PI	0	0 STABLE
-60.0/256 PI	-5.421E-01	-7.363E-01	-7.015E-01	-4.940E-01	-25.0/256 PI	0	0 STABLE
-60.0/256 PI	-5.421E-01	-7.363E-01	-7.161E-01	-4.785E-01	-24.0/256 PI	0	0 STABLE
-60.0/256 PI	-5.421E-01	-7.363E-01	-7.303E-01	-4.624E-01	-23.0/256 PI	0	0 STABLE
-60.0/256 PI	-5.421E-01	-7.363E-01	-7.440E-01	-4.459E-01	-22.0/256 PI	0	0 STABLE
-60.0/256 PI	-5.421E-01	-7.363E-01	-7.572E-01	-4.289E-01	-21.0/256 PI	0	0 STABLE
-60.0/256 PI	-5.421E-01	-7.363E-01	-7.700E-01	-4.115E-01	-20.0/256 PI	0	0 STABLE
-60.0/256 PI	-5.421E-01	-7.363E-01	-7.822E-01	-3.937E-01	-19.0/256 PI	0	0 STABLE
-60.0/256 PI	-5.421E-01	-7.363E-01	-7.938E-01	-3.754E-01	-18.0/256 PI	0	0 STABLE
-60.0/256 PI	-5.421E-01	-7.363E-01	-8.050E-01	-3.568E-01	-17.0/256 PI	0	0 STABLE
-60.0/256 PI	-5.421E-01	-7.363E-01	-8.155E-01	-3.378E-01	-16.0/256 PI	0	0 STABLE
-60.0/256 PI	-5.421E-01	-7.363E-01	-8.255E-01	-3.184E-01	-15.0/256 PI	0	0 STABLE
-60.0/256 PI	-5.421E-01	-7.363E-01	-8.349E-01	-2.987E-01	-14.0/256 PI	0	0 STABLE

Figure 12. (Continued).

COMPUTER RUN NUMBER 2 SATCHE DATA GENERATION
 THE NUMBER OF DELAYS IS 2
 THE CONSTANTS ARE:
 K= 0.450 TAU1= 1.000

B(1) TO B(M): 1.000 1.000 0.0 0.0 0.0
 T(1) TO T(M): 1.000 2.000 0.0 0.0 0.0

**CHECKING IF F2 OF OMEGA SUB IIMIN THROUGH
 F2 OF OMEGA SUB IIMAX LIES OUTSIDE OR INTERSECTS F1**

OBSERVE THE FOLLOWING CODES BELOW
 N=0 : NOT A REGION OF INTERSECTION
 N=1 : A REGION OF INTERSECTION
 L=-1 : X2 IS TO THE RIGHT OF F1, UNSTABLE CASE
 L=1 : F2 INTERSECTS F1, UNSTABLE CASE
 L=0 : F2 IS OUTSIDE OF F1, STABLE CASE

OMEGA OF F1	RE F1	IM F1	RE F2	IM F2	OMEGA OF F2	N	L
-60.0/256 PI	-5.421E-01	-7.363E-01	-8.437E-01	-2.787E-01	-13.0/256 PI	0	0 STABLE
-59.0/256 PI	-5.242E-01	-7.240E-01	-8.519E-01	-2.584E-01	-12.0/256 PI	0	0 STABLE
-59.0/256 PI	-5.242E-01	-7.240E-01	-8.595E-01	-2.378E-01	-11.0/256 PI	0	0 STABLE
-59.0/256 PI	-5.242E-01	-7.240E-01	-8.665E-01	-2.170E-01	-10.0/256 PI	0	0 STABLE
-58.0/256 PI	-5.066E-01	-7.118E-01	-8.728E-01	-1.960E-01	-9.0/256 PI	0	0 STABLE
-58.0/256 PI	-5.066E-01	-7.118E-01	-8.785E-01	-1.747E-01	-8.0/256 PI	0	0 STABLE
-57.0/256 PI	-4.893E-01	-6.995E-01	-8.835E-01	-1.533E-01	-7.0/256 PI	0	0 STABLE
-57.0/256 PI	-4.893E-01	-6.995E-01	-8.879E-01	-1.317E-01	-6.0/256 PI	0	0 STABLE
-56.0/256 PI	-4.723E-01	-6.872E-01	-8.916E-01	-1.099E-01	-5.0/256 PI	0	0 STABLE
-55.0/256 PI	-4.555E-01	-6.749E-01	-8.946E-01	-8.809E-02	-4.0/256 PI	0	0 STABLE
-54.0/256 PI	-4.391E-01	-6.627E-01	-8.970E-01	-6.614E-02	-3.0/256 PI	0	0 STABLE
-54.0/256 PI	-4.391E-01	-6.627E-01	-8.986E-01	-4.413E-02	-2.0/256 PI	0	0 STABLE
-53.0/256 PI	-4.230E-01	-6.504E-01	-8.997E-01	-2.206E-02	-1.0/256 PI	0	0 STABLE
52.0/256 PI	-4.072E-01	6.381E-01	-9.000E-01	2.159E-05	0.0/256 PI	0	0 STABLE

Figure 12. (Continued).

COMPUTER RUN NUMBER 2 SATCHE DATA GENERATION
 THE NUMBER OF DELAYS IS 2
 THE CONSTANTS ARE:
 K= 0.450 TAU1= 1.000

B(1) TO B(M): 1.000 1.000 0.0 0.0 0.0
 T(1) TO T(M): 1.000 2.000 0.0 0.0 0.0

**CHECKING IF F2 OF OMEGA SUB IIMIN THROUGH
 F2 OF OMEGA SUP IIMAX LIES OUTSIDE OR INTERSECTS F1**

OBSERVE THE FOLLOWING CODES BELOW
 N=0 : NOT A REGION OF INTERSECTION
 N=1 : A REGION OF INTERSECTION
 L=-1 : X2 IS TO THE RIGHT OF F1, UNSTABLE CASE
 L=1 : F2 INTERSECTS F1, UNSTABLE CASE
 L=0 : F2 IS OUTSIDE OF F1, STABLE CASE

OMEGA OF F1	RE F1	IM F1	RE F2	IM F2	OMEGA OF F2	N	L
53.0/256 PI	-4.230E-01	6.504E-01	-8.997E-01	2.211E-02	1.0/256 PI	0	0 STABLE
54.0/256 PI	-4.392E-01	6.627E-01	-8.986E-01	4.417E-02	2.0/256 PI	0	0 STABLE
54.0/256 PI	-4.392E-01	6.627E-01	-8.970E-01	6.618E-02	3.0/256 PI	0	0 STABLE
55.0/256 PI	-4.556E-01	6.750E-01	-8.946E-01	8.813E-02	4.0/256 PI	0	0 STABLE
56.0/256 PI	-4.723E-01	6.872E-01	-8.915E-01	1.100E-01	5.0/256 PI	0	0 STABLE
57.0/256 PI	-4.893E-01	6.995E-01	-8.878E-01	1.317E-01	6.0/256 PI	0	0 STABLE
57.0/256 PI	-4.893E-01	6.995E-01	-8.835E-01	1.533E-01	7.0/256 PI	0	0 STABLE
58.0/256 PI	-5.066E-01	7.118E-01	-8.785E-01	1.748E-01	8.0/256 PI	0	0 STABLE
58.0/256 PI	-5.066E-01	7.118E-01	-8.728E-01	1.960E-01	9.0/256 PI	0	0 STABLE
59.0/256 PI	-5.242E-01	7.240E-01	-8.665E-01	2.171E-01	10.0/256 PI	0	0 STABLE
59.0/256 PI	-5.242E-01	7.240E-01	-8.595E-01	2.379E-01	11.0/256 PI	0	0 STABLE
59.0/256 PI	-5.242E-01	7.240E-01	-8.519E-01	2.584E-01	12.0/256 PI	0	0 STABLE
60.0/256 PI	-5.422E-01	7.363E-01	-8.437E-01	2.787E-01	13.0/256 PI	0	0 STABLE
60.0/256 PI	-5.422E-01	7.363E-01	-8.349E-01	2.986E-01	14.0/256 PI	0	0 STABLE

Figure 12. (Continued).

COMPUTER RUN NUMBER 2 SATCHE DATA GENERATION
 THE NUMBER OF DELAYS IS 2
 THE CONSTANTS ARE:
 K= 0.450 TAU1= 1.000

B(1) TO B(M): 1.000 1.000 0.0 0.0 0.0
 T(1) TO T(M): 1.000 2.000 0.0 0.0 0.0

**CHECKING IF F2 OF OMEGA SUB IIMIN THROUGH
 F2 OF OMEGA SUB IIMAX LIES OUTSIDE OR INTERSECTS F1**

OBSERVE THE FOLLOWING CODES BELOW
 N=0 : NOT A REGION OF INTERSECTION
 N=1 : A REGION OF INTERSECTION
 L=-1 : X2 IS TO THE RIGHT OF F1, UNSTABLE CASE
 L=1 : F2 INTERSECTS F1, UNSTABLE CASE
 L=0 : F2 IS OUTSIDE OF F1, STABLE CASE

OMEGA OF F1	RE F1	IM F1	RE F2	IM F2	OMEGA OF F2	N	L
60.0/256 PI	-5.422E-01	7.363E-01	-8.255E-01	3.185E-01	15.0/256 PI	0	0 STABLE
60.0/256 PI	-5.422E-01	7.363E-01	-8.155E-01	3.378E-01	16.0/256 PI	0	0 STABLE
60.0/256 PI	-5.422E-01	7.363E-01	-8.049E-01	3.568E-01	17.0/256 PI	0	0 STABLE
60.0/256 PI	-5.422E-01	7.363E-01	-7.938E-01	3.755E-01	18.0/256 PI	0	0 STABLE
60.0/256 PI	-5.422E-01	7.363E-01	-7.821E-01	3.937E-01	19.0/256 PI	0	0 STABLE
60.0/256 PI	-5.422E-01	7.363E-01	-7.699E-01	4.116E-01	20.0/256 PI	0	0 STABLE
60.0/256 PI	-5.422E-01	7.363E-01	-7.572E-01	4.290E-01	21.0/256 PI	0	0 STABLE
60.0/256 PI	-5.422E-01	7.363E-01	-7.440E-01	4.459E-01	22.0/256 PI	0	0 STABLE
60.0/256 PI	-5.422E-01	7.363E-01	-7.303E-01	4.625E-01	23.0/256 PI	0	0 STABLE
60.0/256 PI	-5.422E-01	7.363E-01	-7.161E-01	4.785E-01	24.0/256 PI	0	0 STABLE
60.0/256 PI	-5.422E-01	7.363E-01	-7.015E-01	4.940E-01	25.0/256 PI	0	0 STABLE
60.0/256 PI	-5.422E-01	7.363E-01	-6.864E-01	5.091E-01	26.0/256 PI	0	0 STABLE
59.0/256 PI	-5.242E-01	7.240E-01	-6.709E-01	5.236E-01	27.0/256 PI	0	0 STABLE
59.0/256 PI	-5.242E-01	7.240E-01	-6.550E-01	5.376E-01	28.0/256 PI	0	0 STABLE

Figure 12. (Continued).

COMPUTER RUN NUMBER 2 SATCHE DATA GENERATION
 THE NUMBER OF DELAYS IS 2
 THE CONSTANTS ARE:
 K= 0.450 TAU1= 1.000

B(1) TO B(M): 1.000 1.000 0.0 0.0 0.0
 T(1) TO T(M): 1.000 2.000 0.0 0.0 0.0

**CHECKING IF F2 OF OMEGA SUB IIMIN THROUGH
 F2 OF OMEGA SUB IIMAX LIES OUTSIDE OR INTERSECTS F1**

OBSERVE THE FOLLOWING CODES BELOW
 N=0 : NOT A REGION OF INTERSECTION
 N=1 : A REGION OF INTERSECTION
 L=-1 : X2 IS TO THE RIGHT OF F1, UNSTABLE CASE
 L=1 : F2 INTERSECTS F1, UNSTABLE CASE
 L=0 : F2 IS OUTSIDE OF F1, STABLE CASE

OMEGA OF F1	RE F1	IM F1	RE F2	IM F2	OMEGA OF F2	N	L
59.0/256 PI	-5.242E-01	7.240E-01	-6.388E-01	5.510E-01	29.0/256 PI	0	0 STABLE
59.0/256 PI	-5.066E-01	7.118E-01	-6.222E-01	5.639E-01	30.0/256 PI	0	0 STABLE
58.0/256 PI	-5.066E-01	7.118E-01	-6.052E-01	5.752E-01	31.0/256 PI	0	0 STABLE
58.0/256 PI	-5.066E-01	7.118E-01	-5.879E-01	5.880E-01	32.0/256 PI	0	0 STABLE
57.0/256 PI	-4.893E-01	6.995E-01	-5.704E-01	5.991E-01	33.0/256 PI	0	0 STABLE
57.0/256 PI	-4.893E-01	6.995E-01	-5.525E-01	6.097E-01	34.0/256 PI	0	0 STABLE
57.0/256 PI	-4.893E-01	6.995E-01	-5.344E-01	6.196E-01	35.0/256 PI	0	0 STABLE
56.0/256 PI	-4.723E-01	6.872E-01	-5.161E-01	6.289E-01	36.0/256 PI	0	0 STABLE
56.0/256 PI	-4.723E-01	6.872E-01	-4.976E-01	6.376E-01	37.0/256 PI	0	0 STABLE
55.0/256 PI	-4.556E-01	6.750E-01	-4.789E-01	6.457E-01	38.0/256 PI	0	0 STABLE
55.0/256 PI	-4.556E-01	6.750E-01	-4.600E-01	6.532E-01	39.0/256 PI	1	1 UNSTABLE
54.0/256 PI	-4.392E-01	6.627E-01	-4.410E-01	6.600E-01	40.0/256 PI	1	1 UNSTABLE
53.0/256 PI	-4.230E-01	6.504E-01	-4.218E-01	6.661E-01	41.0/256 PI	1	1 UNSTABLE
53.0/256 PI	-4.230E-01	6.504E-01	-4.026E-01	6.717E-01	42.0/256 PI	0	-1 UNSTABLE

Figure 12. (Continued).

COMPUTER RUN NUMBER 2 SATCHE DATA GENERATION
 THE NUMBER OF DELAYS IS 2
 THE CONSTANTS ARE:
 K= 0.450 TAU1= 1.000

B(1) TO B(M): 1.000 1.000 0.0 0.0 0.0
 T(1) TO T(M): 1.000 2.000 0.0 0.0 0.0

**CHECKING IF F2 OF OMEGA SUB IIMIN THROUGH
 F2 OF OMEGA SUB IIMAX LIES OUTSIDE OR INTERSECTS F1**

OBSERVE THE FOLLOWING CODES BELOW
 N=0 : NOT A REGION OF INTERSECTION
 N=1 : A REGION OF INTERSECTION
 L=-1 : X2 IS TO THE RIGHT OF F1, UNSTABLE CASE
 L=1 : F2 INTERSECTS F1, UNSTABLE CASE
 L=0 : F2 IS OUTSIDE OF F1, STABLE CASE

OMEGA OF F1	RE F1	IM F1	RE F2	IM F2	OMEGA OF F2	N	L
52.0/256 PI	-4.072E-01	6.381E-01	-3.832E-01	6.756E-01	43.0/256 PI	0	-1 UNSTABLE
52.0/256 PI	-4.072E-01	6.381E-01	-3.639E-01	6.808E-01	44.0/256 PI	0	-1 UNSTABLE
51.0/256 PI	-3.917E-01	6.259E-01	-3.445E-01	6.844E-01	45.0/256 PI	0	-1 UNSTABLE
50.0/256 PI	-3.765E-01	6.136E-01	-3.251E-01	6.874E-01	46.0/256 PI	0	-1 UNSTABLE
50.0/256 PI	-3.765E-01	6.136E-01	-3.057E-01	6.897E-01	47.0/256 PI	0	-1 UNSTABLE
49.0/256 PI	-3.616E-01	6.013E-01	-2.864E-01	6.914E-01	48.0/256 PI	0	-1 UNSTABLE
48.0/256 PI	-3.470E-01	5.891E-01	-2.671E-01	6.924E-01	49.0/256 PI	0	-1 UNSTABLE
48.0/256 PI	-3.470E-01	5.891E-01	-2.479E-01	6.928E-01	50.0/256 PI	0	-1 UNSTABLE
47.0/256 PI	-3.327E-01	5.768E-01	-2.288E-01	6.926E-01	51.0/256 PI	0	-1 UNSTABLE
46.0/256 PI	-3.187E-01	5.645E-01	-2.098E-01	6.918E-01	52.0/256 PI	0	-1 UNSTABLE
45.0/256 PI	-3.050E-01	5.522E-01	-1.910E-01	6.903E-01	53.0/256 PI	0	-1 UNSTABLE
45.0/256 PI	-3.050E-01	5.522E-01	-1.724E-01	6.892E-01	54.0/256 PI	0	-1 UNSTABLE
44.0/256 PI	-2.916E-01	5.400E-01	-1.539E-01	6.856E-01	55.0/256 PI	0	-1 UNSTABLE

Figure 12. (Concluded).

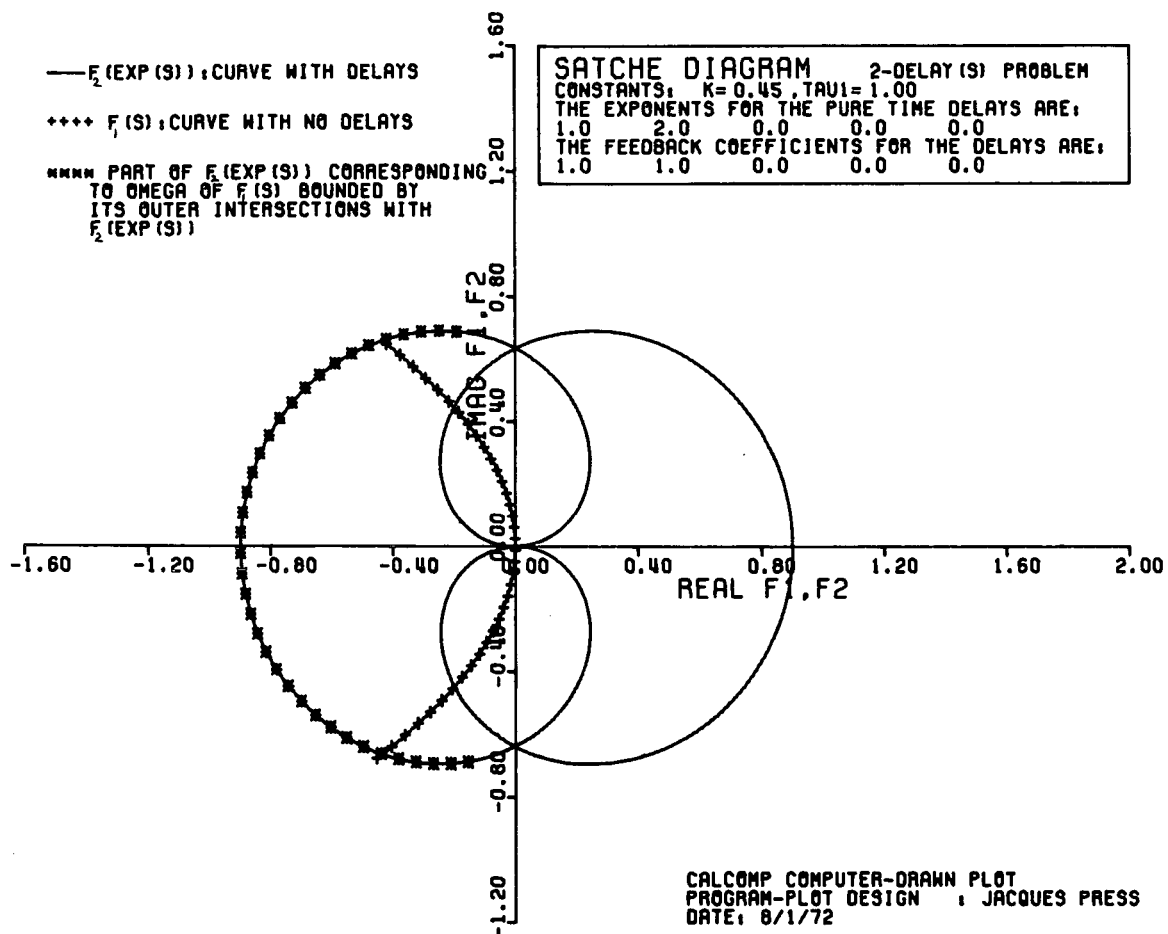


Figure 13. First Satche diagram for a case with two delays.

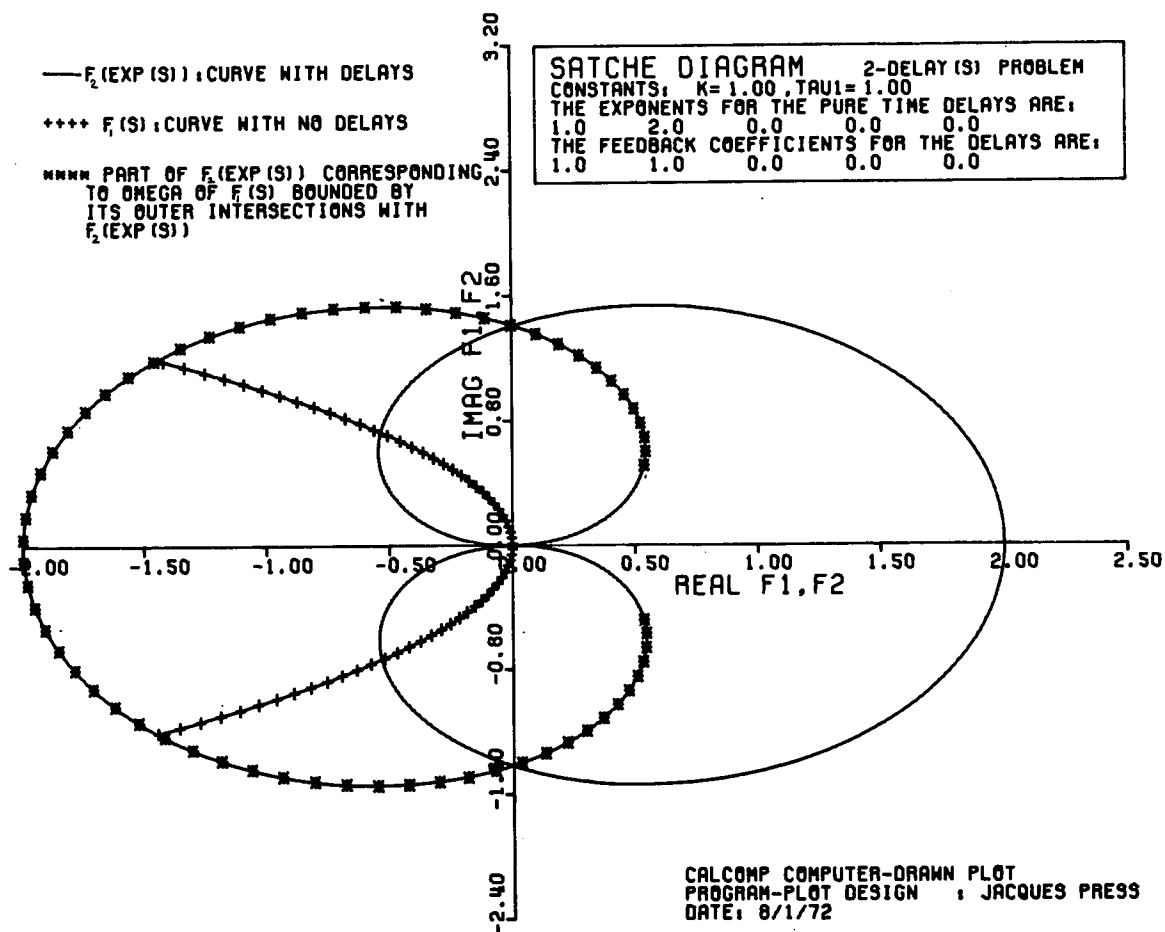


Figure 14. Second Satche diagram for a case with two delays.

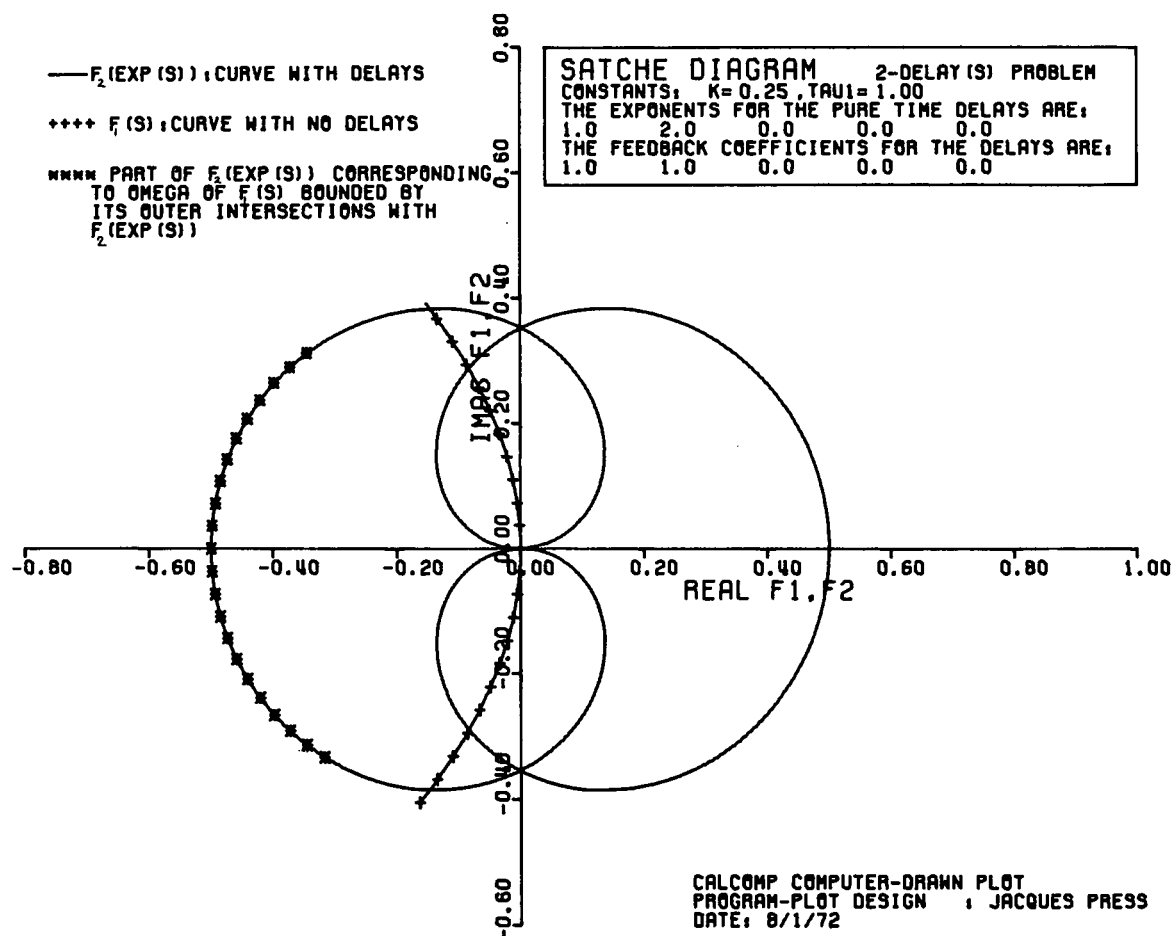


Figure 15. Third Satche diagram for a case with two delays.

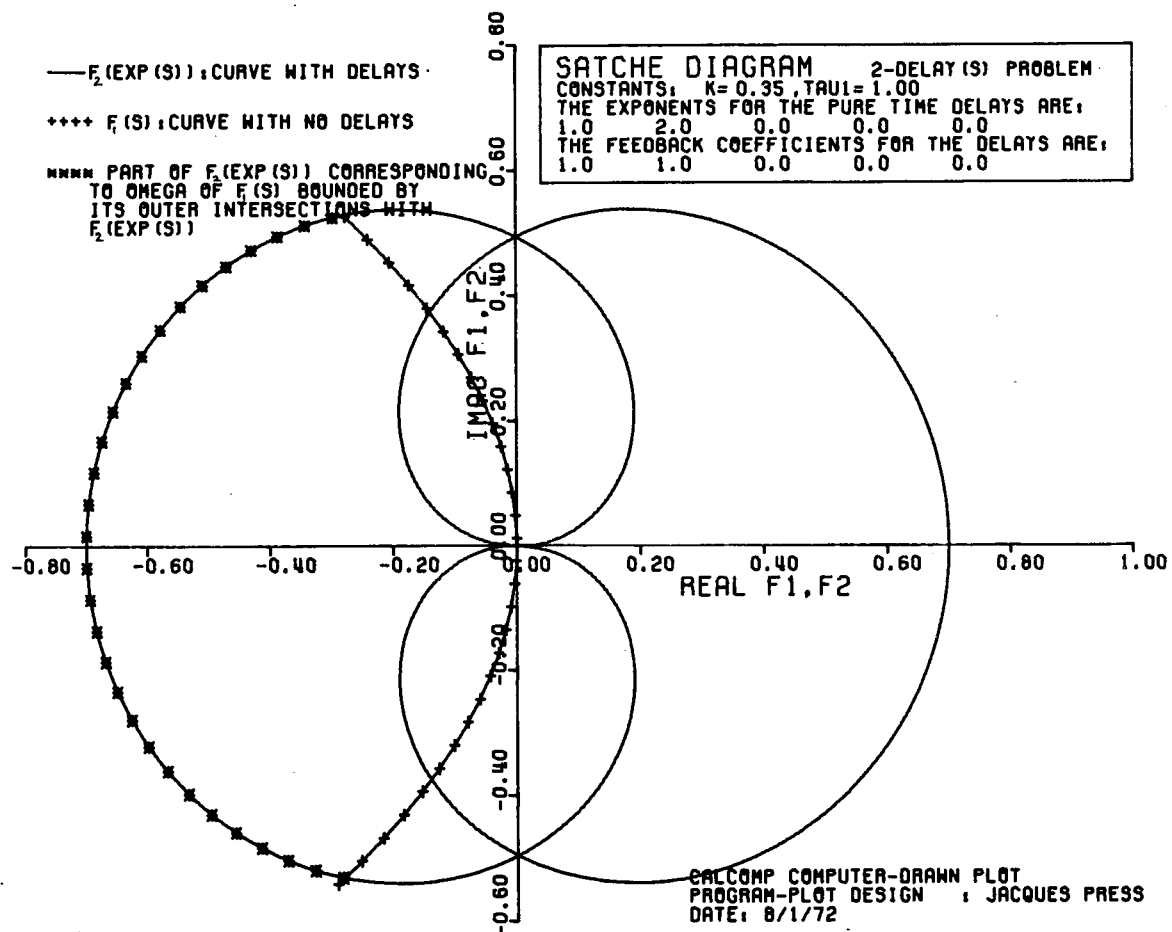


Figure 16. Fourth Satche diagram for a case with two delays.

****CONTINUOUS SYSTEM MODELING PROGRAM****

*** VERSION 1.3 ***

```

LABEL    TIME DELAY MODEL UNIT STEP INPUT
CONSTANT B1=1. ,B2=1.
PARAMETER XK=0.45, TAU1=1. . . .
          T1=1. , T2=2.

DYNAMIC
X1=STEP(0.)
X2=X1-B1*X6-B2*X7
X3=XK*X2
X4=REALPL(0. ,TAU1,X3 )
X5=INTGRL(0. ,X4)
X6=DELAY(10,T1,X5)
X7=DELAY(10,T2,X6)
TIMER DELT=0.05,FINTIM=80. ,PRODEL=1.0 ,OUTDEL=1.0
METHOD ADAMS
PRTPLT X7
END
PARAMETER XK=0.25      (PROGRAM IS RERUN AT THIS STEP)
END
PARAMETER XK=0.35      (PROGRAM IS RERUN AT THIS STEP)
END
STOP

```

OUTPUT VARIABLE SEQUENCE

X6	X7	X1	X2	X3	ZZ0003	X4	X5
OUTPUTS	INPUTS	PARAMS	INTEGS	+	MEM BLKS	FORTRAN	DATA CDS
12(500)	38(1400)	10(400)	2+	2=	4(300)	2(500)	12

ENDJOB

NOTATION:

XK=K (AS IN TEXT)

REALPL, INTGRL, DELAY, STEP = FUNCTIONS DESCRIBED IN APPENDIX F.

METHOD ADAMS = INTEGRATION METHOD USED. (SEE REF. 52)

TIMER = DESCRIBES THE SELECTED TIMING SEQUENCE.

Figure 17. Continuous System Modeling Program (simulation) for the case of two delays.

TIME DELAY MODEL UNIT STEP INPUT

PAGE 1

TIME	X7	MINIMUM	X7	VERSUS TIME	MAXIMUM
		-1.2101E 01 I			1.0575E 01 I
0.0	0.0	-----+			
1.0000E 00	0.0	-----+			
2.0000E 00	0.0	-----+			
3.0000E 00	0.0	-----+			
4.0000E 00	1.6572E-01	-----+			
5.0000E 00	5.0530E-01	-----+			
6.0000E 00	8.5712E-01	-----+			
7.0000E 00	1.1128E 00	-----+			
8.0000E 00	1.1849E 00	-----+			
9.0000E 00	1.0273E 00	-----+			
1.0000E 01	6.7453E-01	-----+			
1.1000E 01	2.3200E-01	-----+			
1.2000E 01	-1.5673E-01	-----+			
1.3000E 01	-3.5369E-01	-----+			
1.4000E 01	-2.7464E-01	-----+			
1.5000E 01	7.7014E-02	-----+			
1.6000E 01	6.0409E-01	-----+			
1.7000E 01	1.1408E 00	-----+			
1.8000E 01	1.5039E 00	-----+			
1.9000E 01	1.5538E 00	-----+			
2.0000E 01	1.2464E 00	-----+			
2.1000E 01	6.5637E-01	-----+			
2.2000E 01	-3.9105E-02	-----+			
2.3000E 01	-6.0776E-01	-----+			
2.4000E 01	-8.4637E-01	-----+			
2.5000E 01	-6.4355E-01	-----+			
2.6000E 01	-3.0878E-02	-----+			
2.7000E 01	8.1673E-01	-----+			
2.8000E 01	1.6286E 00	-----+			
2.9000E 01	2.1232E 00	-----+			
3.0000E 01	2.1034E 00	-----+			
3.1000E 01	1.5317E 00	-----+			
3.2000E 01	5.5676E-01	-----+			
3.3000E 01	-5.2163E-01	-----+			
3.4000E 01	-1.3424E 00	-----+			
3.5000E 01	-1.6016E 00	-----+			
3.6000E 01	-1.1620E 00	-----+			
3.7000E 01	-1.1433E-01	-----+			
3.8000E 01	1.2344E 00	-----+			
3.9000E 01	2.4473E 00	-----+			
4.0000E 01	3.0965E 00	-----+			
4.1000E 01	2.9106E 00	-----+			
4.2000E 01	1.8816E 00	-----+			
4.3000E 01	2.8990E-01	-----+			
4.4000E 01	-1.3669E 00	-----+			
4.5000E 01	-2.5256E 00	-----+			
4.6000E 01	-2.7461E 00	-----+			
4.7000E 01	-1.8717E 00	-----+			
4.8000E 01	-1.0819E-01	-----+			
4.9000E 01	2.0159E 00	-----+			
5.0000E 01	3.8029E 00	-----+			

Figure 18. Output to Figure 17 with $K = 0.45$.

TIME DELAY MODEL UNIT STEP INPUT

PAGE 2

TIME	X7	MINIMUM	X7	VERSUS TIME	MAXIMUM
		-1.2101E 01			1.0575E 01
		I			I
5.1000E 01	4.6103E 00	-----+-----			
5.2000E 01	4.0769E 00	-----+-----			
5.3000E 01	2.2703E 00	-----+-----			
5.4000E 01	-2.5934E-01	-----+-----			
5.5000E 01	-2.3160E 00	-----+-----			
5.6000E 01	-4.4117E 00	-----+-----			
5.7000E 01	-4.4594E 00	-----+-----			
5.8000E 01	-2.8095E 00	-----+-----			
5.9000E 01	1.1811E-01	-----+-----			
6.0000E 01	3.4289E 00	-----+-----			
6.1000E 01	6.0198E 00	-----+-----			
6.2000E 01	6.9399E 00	-----+-----			
6.3000E 01	5.7295E 00	-----+-----			
6.4000E 01	2.6204E 00	-----+-----			
6.5000E 01	-1.4840E 00	-----+-----			
6.6000E 01	-5.2611E 00	-----+-----			
6.7000E 01	-7.3869E 00	-----+-----			
6.8000E 01	-6.9893E 00	-----+-----			
6.9000E 01	-3.9877E 00	-----+-----			
7.0000E 01	8.1219E-01	-----+-----			
7.1000E 01	5.9184E 00	-----+-----			
7.2000E 01	9.6052E 00	-----+-----			
7.3000E 01	1.0485E 01	-----+-----			
7.4000E 01	8.0148E 00	-----+-----			
7.5000E 01	2.7531E 00	-----+-----			
7.6000E 01	-3.7349E 00	-----+-----			
7.7000E 01	-9.3286E 00	-----+-----			
7.8000E 01	-1.2032E 01	-----+-----			
7.9000E 01	-1.0667E 01	-----+-----			
8.0000E 01	-5.3538E 00	-----+-----			

Figure 18. (Continued)

TIME	X7	MINIMUM 0.0	I	VERSUS TIME	MAXIMUM 8.5620E-01 I
0.0	0.0		+		
1.0000E 00	0.0		+		
2.0000E 00	0.0		+		
3.0000E 00	0.0		+		
4.0000E 00	9.2065E-02		-----+		
5.0000E 00	2.8214E-01		-----+-----+		
6.0000E 00	4.9230E-01		-----+-----+-----+		
7.0000E 00	6.7832E-01		-----+-----+-----+-----+		
8.0000E 00	8.0689E-01		-----+-----+-----+-----+-----+		
9.0000E 00	8.5620E-01		-----+-----+-----+-----+-----+-----+		
1.0000E 01	8.2467E-01		-----+-----+-----+-----+-----+-----+		
1.1000E 01	7.2930E-01		-----+-----+-----+-----+-----+-----+		
1.2000E 01	5.9470E-01		-----+-----+-----+-----+-----+-----+		
1.3000E 01	4.5623E-01		-----+-----+-----+-----+-----+-----+		
1.4000E 01	3.4264E-01		-----+-----+-----+-----+-----+-----+		
1.5000E 01	2.7509E-01		-----+-----+-----+-----+-----+-----+		
1.6000E 01	2.6257E-01		-----+-----+-----+-----+-----+-----+		
1.7000E 01	3.0153E-01		-----+-----+-----+-----+-----+-----+		
1.8000E 01	3.7794E-01		-----+-----+-----+-----+-----+-----+		
1.9000E 01	4.7144E-01		-----+-----+-----+-----+-----+-----+		
2.0000E 01	5.6039E-01		-----+-----+-----+-----+-----+-----+		
2.1000E 01	6.2655E-01		-----+-----+-----+-----+-----+-----+		
2.2000E 01	6.5855E-01		-----+-----+-----+-----+-----+-----+		
2.3000E 01	6.5350E-01		-----+-----+-----+-----+-----+-----+		
2.4000E 01	6.1666E-01		-----+-----+-----+-----+-----+-----+		
2.5000E 01	5.5939E-01		-----+-----+-----+-----+-----+-----+		
2.6000E 01	4.9607E-01		-----+-----+-----+-----+-----+-----+		
2.7000E 01	4.4074E-01		-----+-----+-----+-----+-----+-----+		
2.8000E 01	4.0416E-01		-----+-----+-----+-----+-----+-----+		
2.9000E 01	3.9192E-01		-----+-----+-----+-----+-----+-----+		
3.0000E 01	4.0385E-01		-----+-----+-----+-----+-----+-----+		
3.1000E 01	4.3470E-01		-----+-----+-----+-----+-----+-----+		
3.2000E 01	4.7584E-01		-----+-----+-----+-----+-----+-----+		
3.3000E 01	5.1741E-01		-----+-----+-----+-----+-----+-----+		
3.4000E 01	5.5062E-01		-----+-----+-----+-----+-----+-----+		
3.5000E 01	5.6940E-01		-----+-----+-----+-----+-----+-----+		
3.6000E 01	5.7142E-01		-----+-----+-----+-----+-----+-----+		
3.7000E 01	5.5816E-01		-----+-----+-----+-----+-----+-----+		
3.8000E 01	5.3413E-01		-----+-----+-----+-----+-----+-----+		
3.9000E 01	5.0563E-01		-----+-----+-----+-----+-----+-----+		
4.0000E 01	4.7916E-01		-----+-----+-----+-----+-----+-----+		
4.1000E 01	4.6007E-01		-----+-----+-----+-----+-----+-----+		
4.2000E 01	4.5157E-01		-----+-----+-----+-----+-----+-----+		
4.3000E 01	4.5423E-01		-----+-----+-----+-----+-----+-----+		
4.4000E 01	4.6625E-01		-----+-----+-----+-----+-----+-----+		
4.5000E 01	4.8402E-01		-----+-----+-----+-----+-----+-----+		
4.6000E 01	5.0315E-01		-----+-----+-----+-----+-----+-----+		
4.7000E 01	5.1945E-01		-----+-----+-----+-----+-----+-----+		
4.8000E 01	5.2981E-01		-----+-----+-----+-----+-----+-----+		
4.9000E 01	5.3271E-01		-----+-----+-----+-----+-----+-----+		
5.0000E 01	5.2839E-01		-----+-----+-----+-----+-----+-----+		

Figure 19. Output to Figure 17 with $K = 0.25$.

TIME DELAY MODEL UNIT STEP INPUT

PAGE 2

TIME	X7	MINIMUM	X7	VERSUS TIME	MAXIMUM
		0.0			I
5.1000E 01	5.1857E-01	-----		-----	+
5.2000E 01	5.0594E-01	-----		-----	+
5.3000E 01	4.9349E-01	-----		-----	+
5.4000E 01	4.8382E-01	-----		-----	+
5.5000E 01	4.7865E-01	-----		-----	+
5.6000E 01	4.7857E-01	-----		-----	+
5.7000E 01	4.8301E-01	-----		-----	+
5.8000E 01	4.9054E-01	-----		-----	+
5.9000E 01	4.9920E-01	-----		-----	+
6.0000E 01	5.0706E-01	-----		-----	+
6.1000E 01	5.1254E-01	-----		-----	+
6.2000E 01	5.1475E-01	-----		-----	+
6.3000E 01	5.1361E-01	-----		-----	+
6.4000E 01	5.0972E-01	-----		-----	+
6.5000E 01	5.0423E-01	-----		-----	+
6.6000E 01	4.9846E-01	-----		-----	+
6.7000E 01	4.9367E-01	-----		-----	+
6.8000E 01	4.9076E-01	-----		-----	+
6.9000E 01	4.9012E-01	-----		-----	+
7.0000E 01	4.9164E-01	-----		-----	+
7.1000E 01	4.9475E-01	-----		-----	+
7.2000E 01	4.9861E-01	-----		-----	+
7.3000E 01	5.0233E-01	-----		-----	+
7.4000E 01	5.0514E-01	-----		-----	+
7.5000E 01	5.0655E-01	-----		-----	+
7.6000E 01	5.0641E-01	-----		-----	+
7.7000E 01	5.0494E-01	-----		-----	+
7.8000E 01	5.0259E-01	-----		-----	+
7.9000E 01	4.9997E-01	-----		-----	+
8.0000E 01	4.9764E-01	-----		-----	+

Figure 19. (Continued)

TIME DELAY MODEL UNIT STEP INPUT

PAGE 1

TIME	X7	MINIMUM -2.1531E-02	X7	VERSUS TIME	MAXIMUM 1.0306E 00
0.0	0.0	I			I
1.0000E 00	0.0	-+			
2.0000E 00	0.0	-+			
3.0000E 00	0.0	-+			
4.0000E 00	1.2889E-01	-----+			
5.0000E 00	3.9400E-01	-----+			
6.0000E 00	6.7793E-01	-----+			
7.0000E 00	9.0743E-01	-----+			
8.0000E 00	1.0245E 00	-----+			
9.0000E 00	9.9422E-01	-----+			
1.0000E 01	8.2593E-01	-----+			
1.1000E 01	5.6764E-01	-----+			
1.2000E 01	2.9262E-01	-----+			
1.3000E 01	7.6824E-02	-----+			
1.4000E 01	-1.9587E-02	+			
1.5000E 01	2.9534E-02	---+			
1.6000E 01	2.0957E-01	-----+			
1.7000E 01	4.6950E-01	-----+			
1.8000E 01	7.3631E-01	-----+			
1.9000E 01	9.3556E-01	-----+			
2.0000E 01	1.0121E 00	-----+			
2.1000E 01	9.4527E-01	-----+			
2.2000E 01	7.5469E-01	-----+			
2.3000E 01	4.9426E-01	-----+			
2.4000E 01	2.3704E-01	-----+			
2.5000E 01	5.4775E-02	---+			
2.6000E 01	-2.1728E-03	+			
2.7000E 01	8.1304E-02	---+			
2.8000E 01	2.9105E-01	-----+			
2.9000E 01	5.4062E-01	-----+			
3.0000E 01	7.8725E-01	-----+			
3.1000E 01	9.5222E-01	-----+			
3.2000E 01	9.9003E-01	-----+			
3.3000E 01	8.9092E-01	-----+			
3.4000E 01	6.8339E-01	-----+			
3.5000E 01	4.2599E-01	-----+			
3.6000E 01	1.9084E-01	-----+			
3.7000E 01	4.3390E-02	---+			
3.8000E 01	2.4215E-02	---+			
3.9000E 01	1.3787E-01	-----+			
4.0000E 01	3.5182E-01	-----+			
4.1000E 01	6.0578E-01	-----+			
4.2000E 01	8.2865E-01	-----+			
4.3000E 01	9.5846E-01	-----+			
4.4000E 01	9.5959E-01	-----+			
4.5000E 01	8.3252E-01	-----+			
4.6000E 01	6.1349E-01	-----+			
4.7000E 01	3.6416E-01	-----+			
4.8000E 01	1.5428E-01	-----+			
4.9000E 01	4.2132E-02	---+			
5.0000E 01	5.8395E-02	---+			

Figure 20. Output to Figure 17 with K =0.35.

TIME DELAY MODEL UNIT STEP INPUT

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TIME	X7	MINIMUM -2.1531E-02 ↑	X7	VERSUS TIME	MAXIMUM 1.0306E 00 ↓
5.1000E 01	1.9773E-01	-----+			
5.2000E 01	4.2051E-01	-----+			
5.3000E 01	6.6408E-01	-----+			
5.4000E 01	8.6037E-01	-----+			
5.5000E 01	9.5491E-01	-----+			
5.6000E 01	9.2198E-01	-----+			
5.7000E 01	7.7156E-01	-----+			
5.8000E 01	5.4632E-01	-----+			
5.9000E 01	3.0956E-01	-----+			
6.0000E 01	1.2740E-01	-----+			
6.1000E 01	5.0297E-02	-----+			
6.2000E 01	9.9109E-02	-----+			
6.3000E 01	2.5944E-01	-----+			
6.4000E 01	4.8587E-01	-----+			
6.5000E 01	7.1483E-01	-----+			
6.6000E 01	8.8245E-01	-----+			
6.7000E 01	9.4235E-01	-----+			
6.8000E 01	8.7849E-01	-----+			
6.9000E 01	7.0946E-01	-----+			
7.0000E 01	4.8305E-01	-----+			
7.1000E 01	2.6278E-01	-----+			
7.2000E 01	1.1005E-01	-----+			
7.3000E 01	6.7037E-02	-----+			
7.4000E 01	1.4506E-01	-----+			
7.5000E 01	3.2159E-01	-----+			
7.6000E 01	5.4679E-01	-----+			
7.7000E 01	7.5754E-01	-----+			
7.8000E 01	8.9514E-01	-----+			
7.9000E 01	9.2167E-01	-----+			
8.0000E 01	8.3042E-01	-----+			

Figure 20. (Continued).

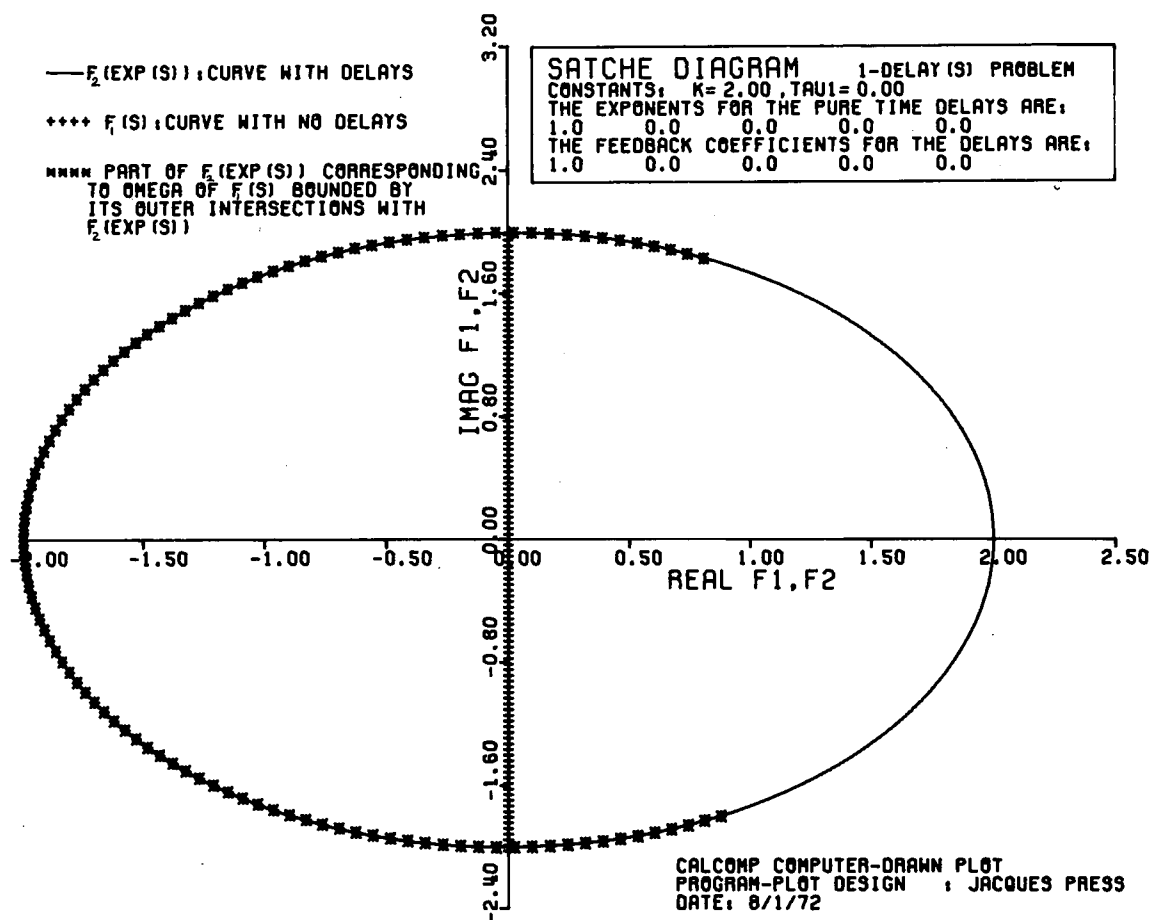


Figure 21. First Satche diagram for a single delay case.

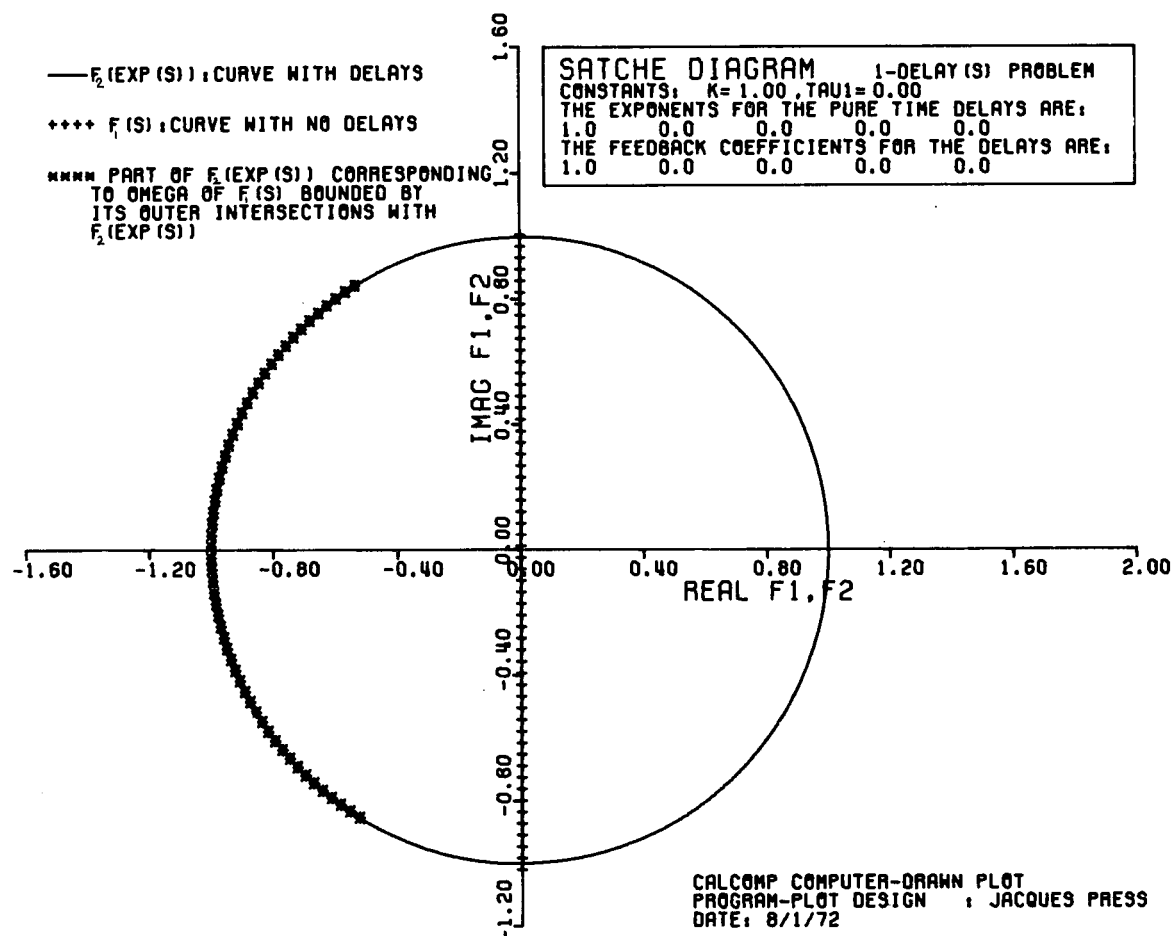


Figure 22. Second Satche diagram for a single delay case.

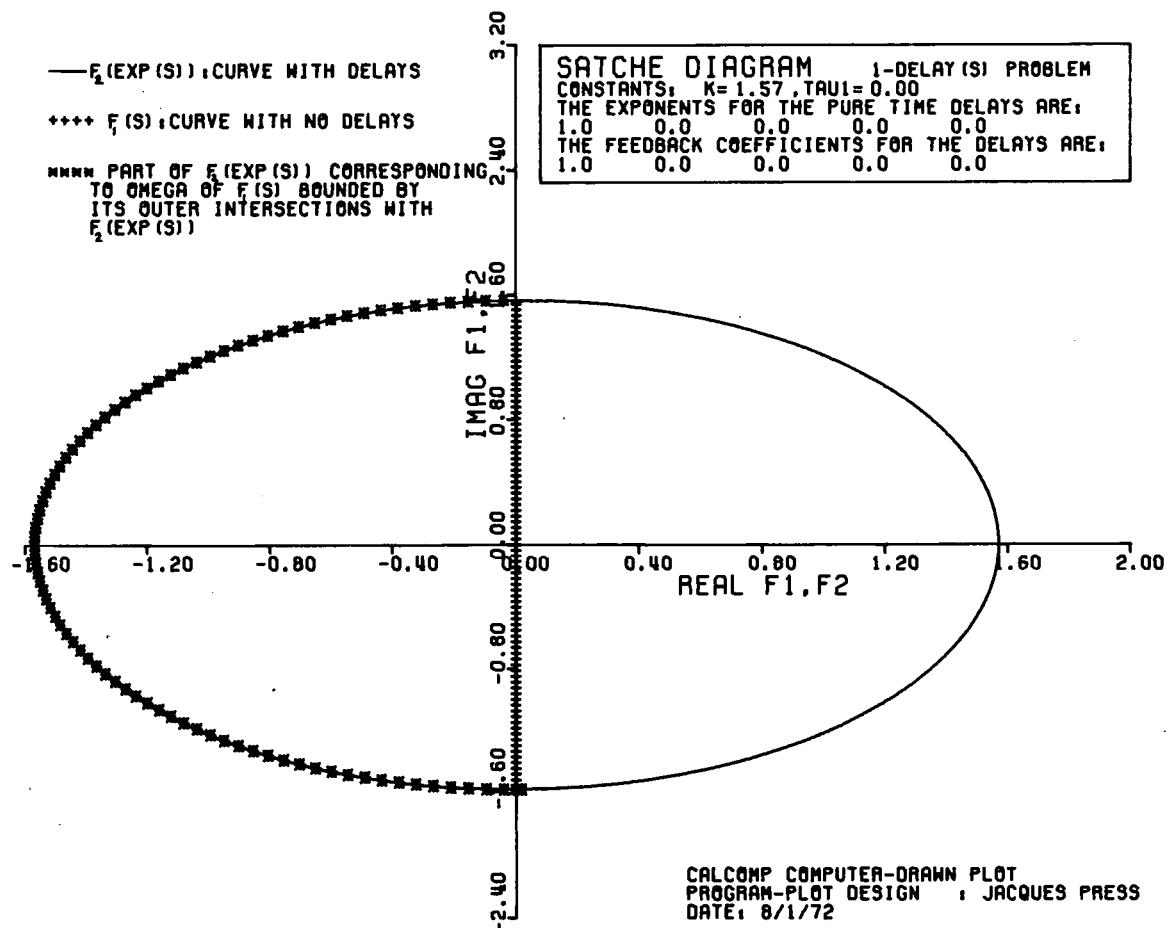


Figure 23. Third Satche diagram for a single delay case.

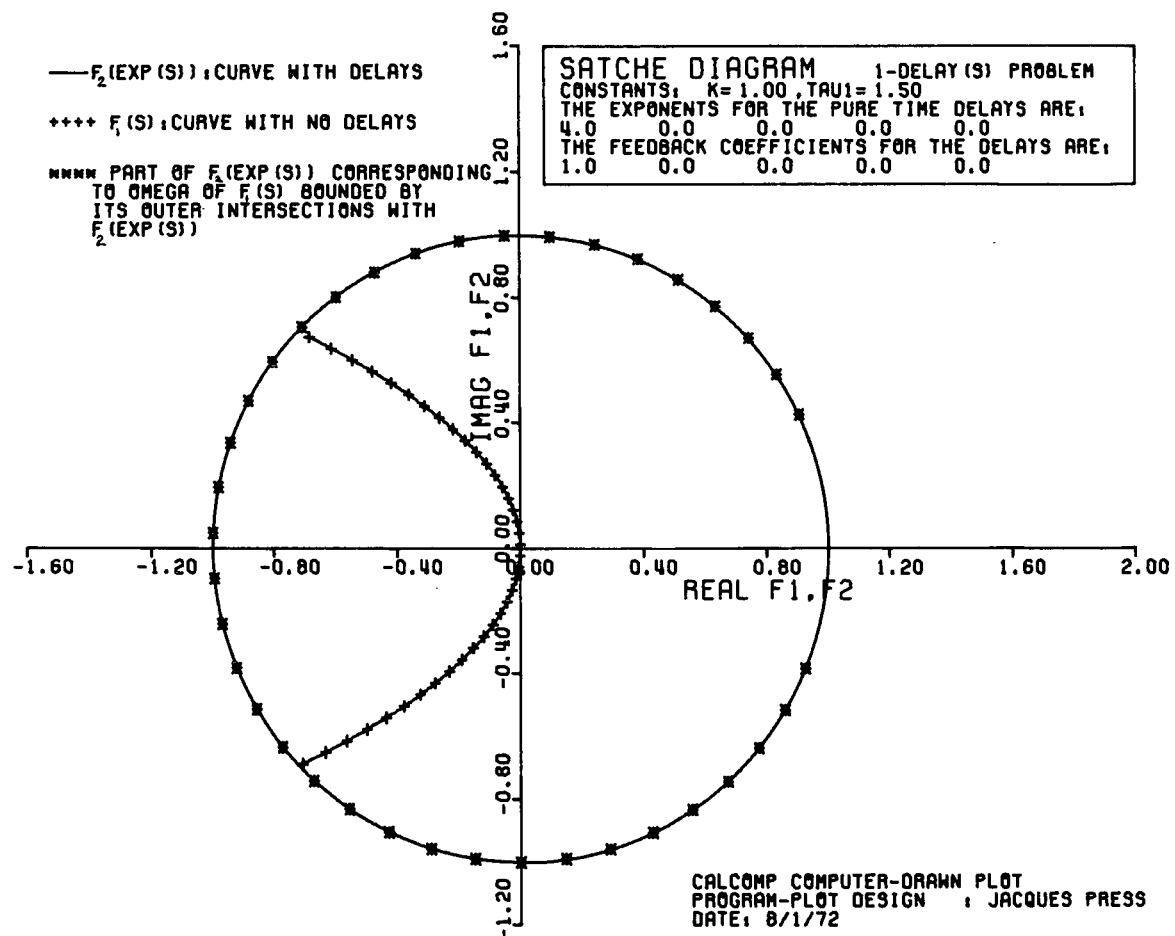


Figure 24. Fourth Satche diagram for a single delay case.

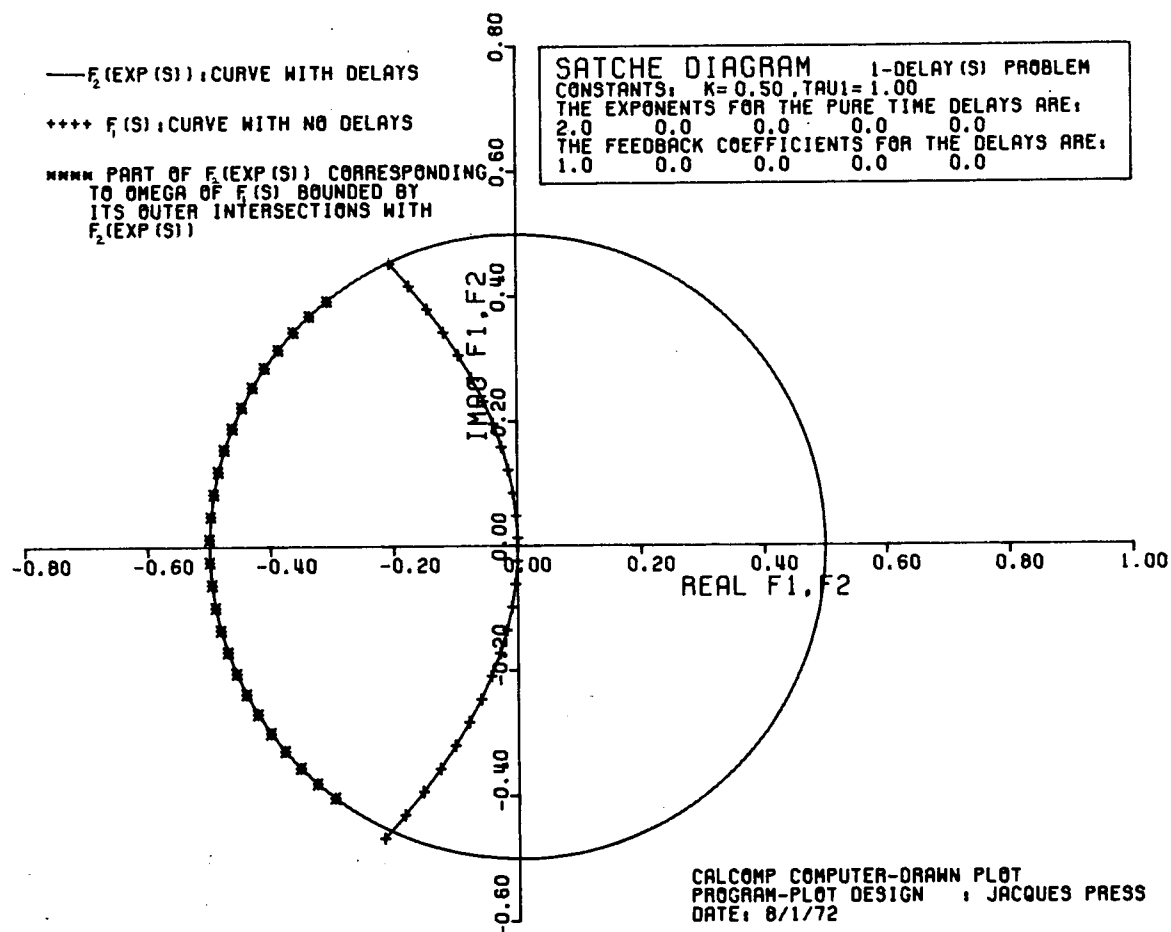


Figure 25. Fifth Satche diagram for a single delay case.

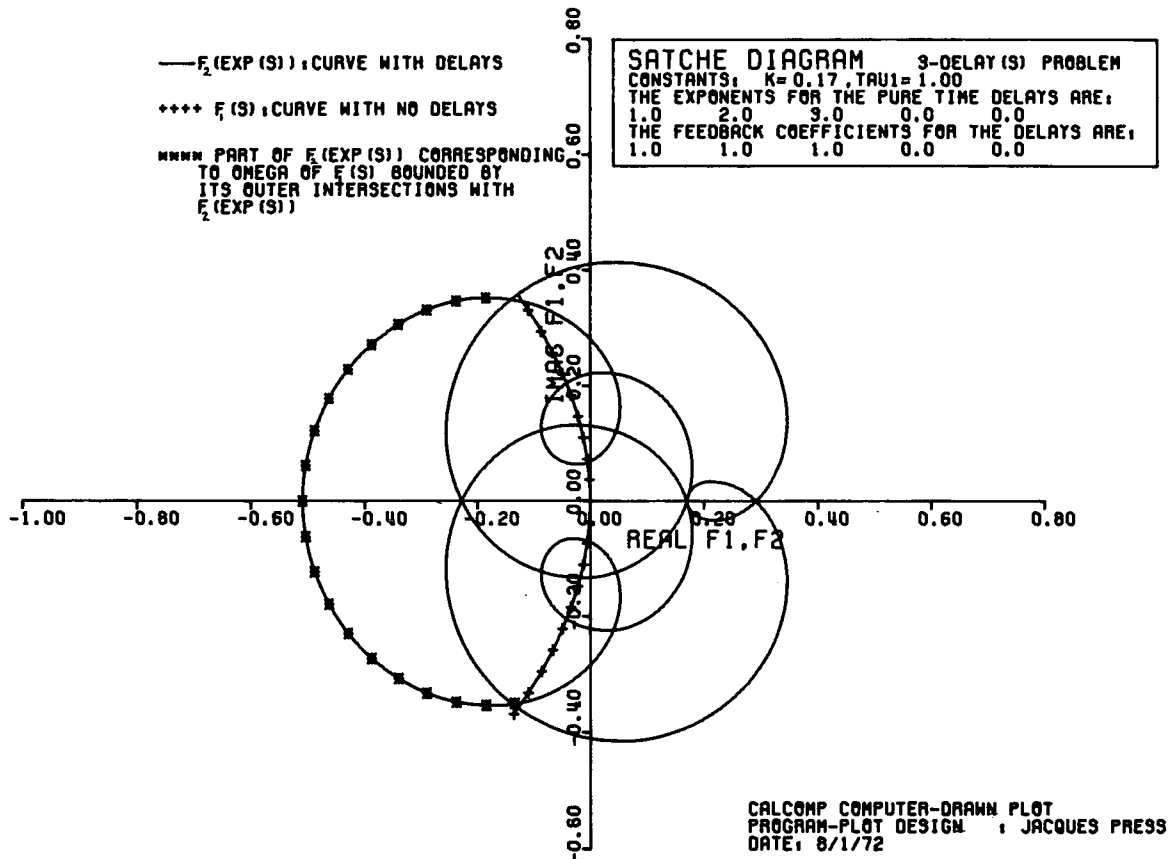


Figure 26. First Satche diagram for a case with three delays.

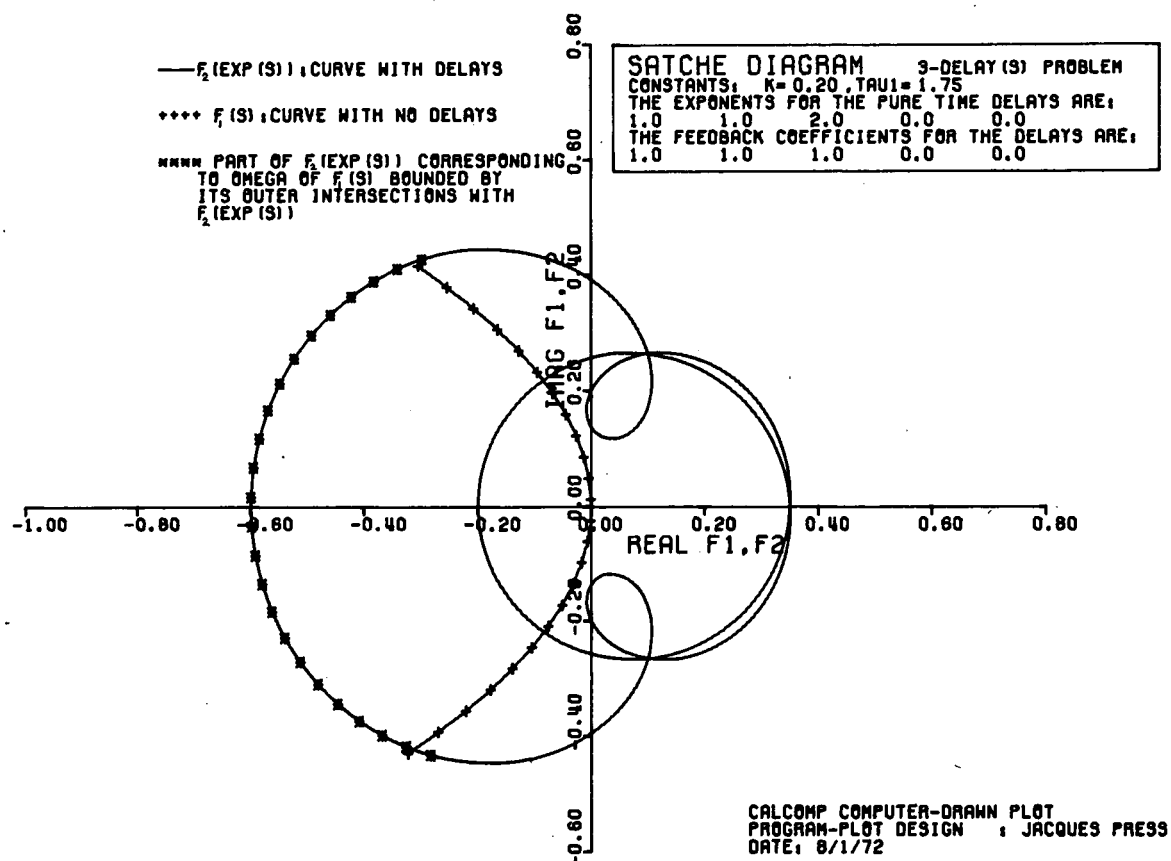


Figure 27. Second Satche diagram for a case with three delays.

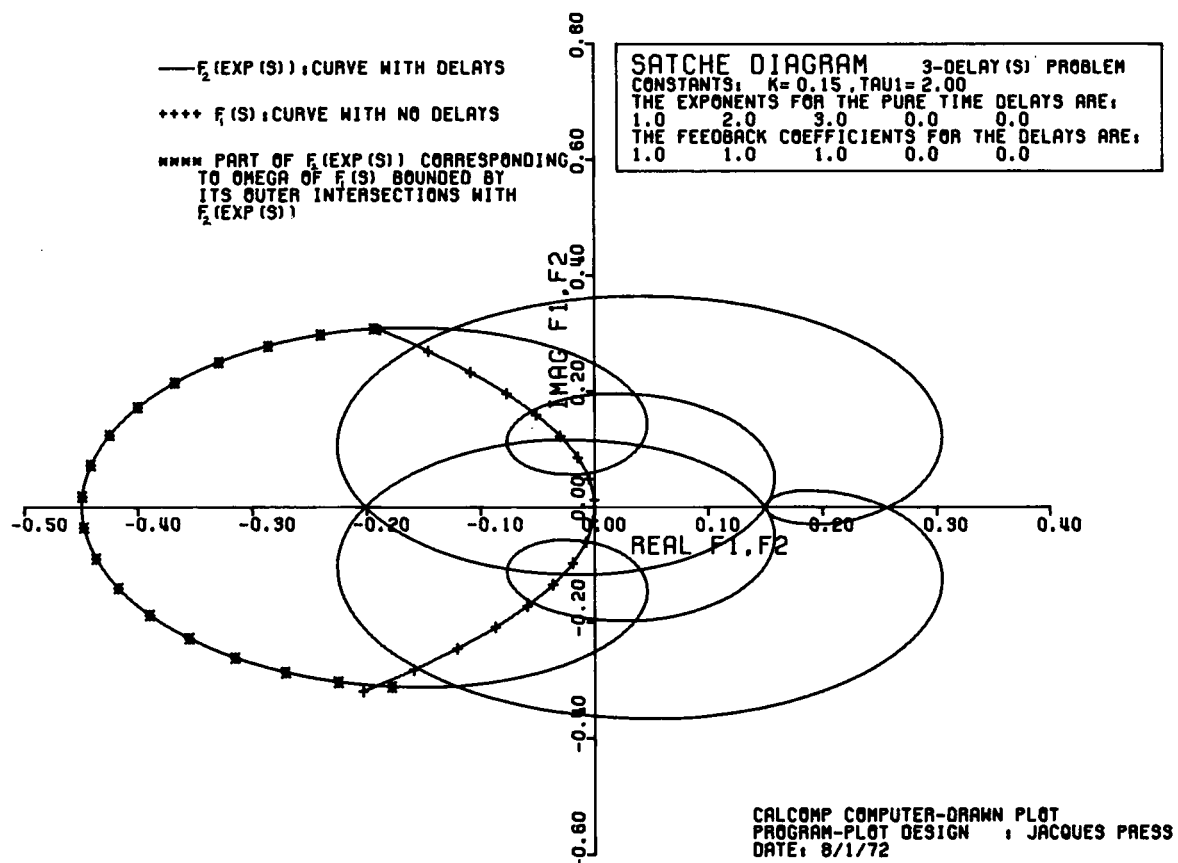


Figure 28. Third Satche diagram for a case with three delays.

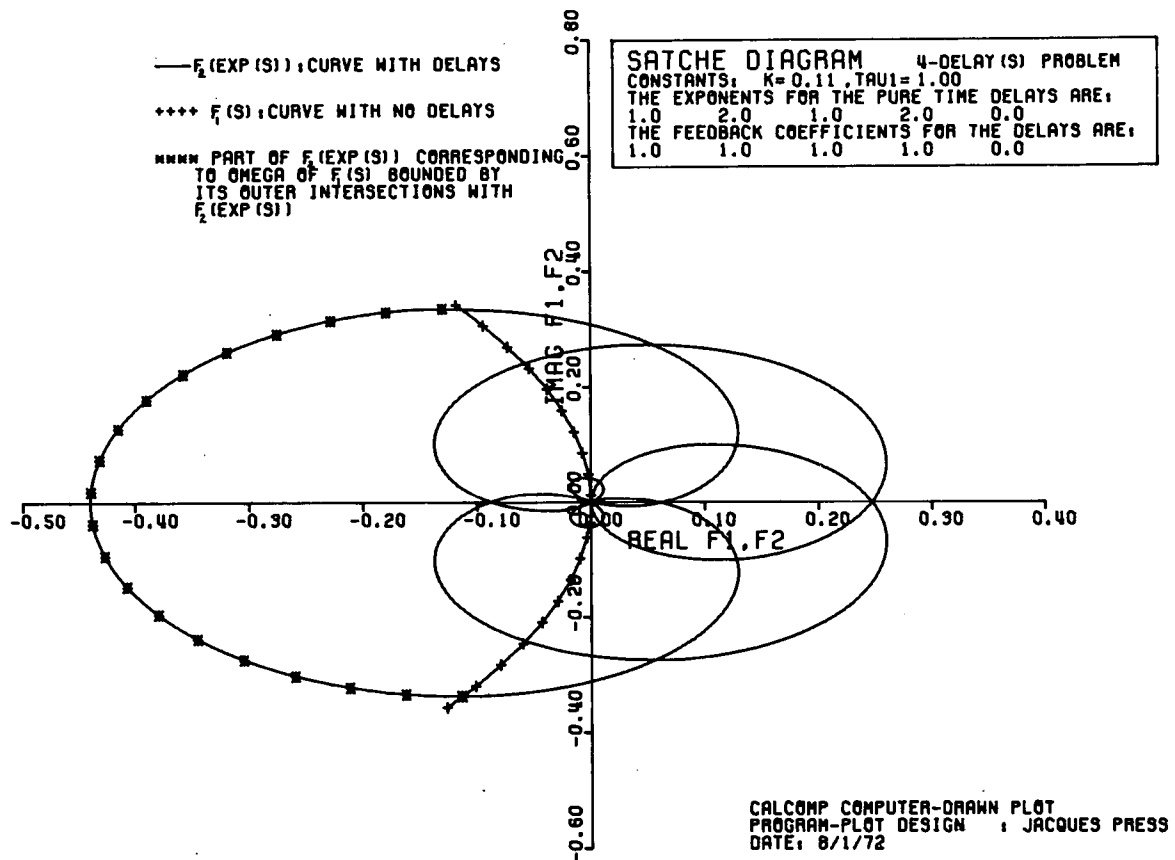


Figure 29. Satche diagram for a case with four delays.

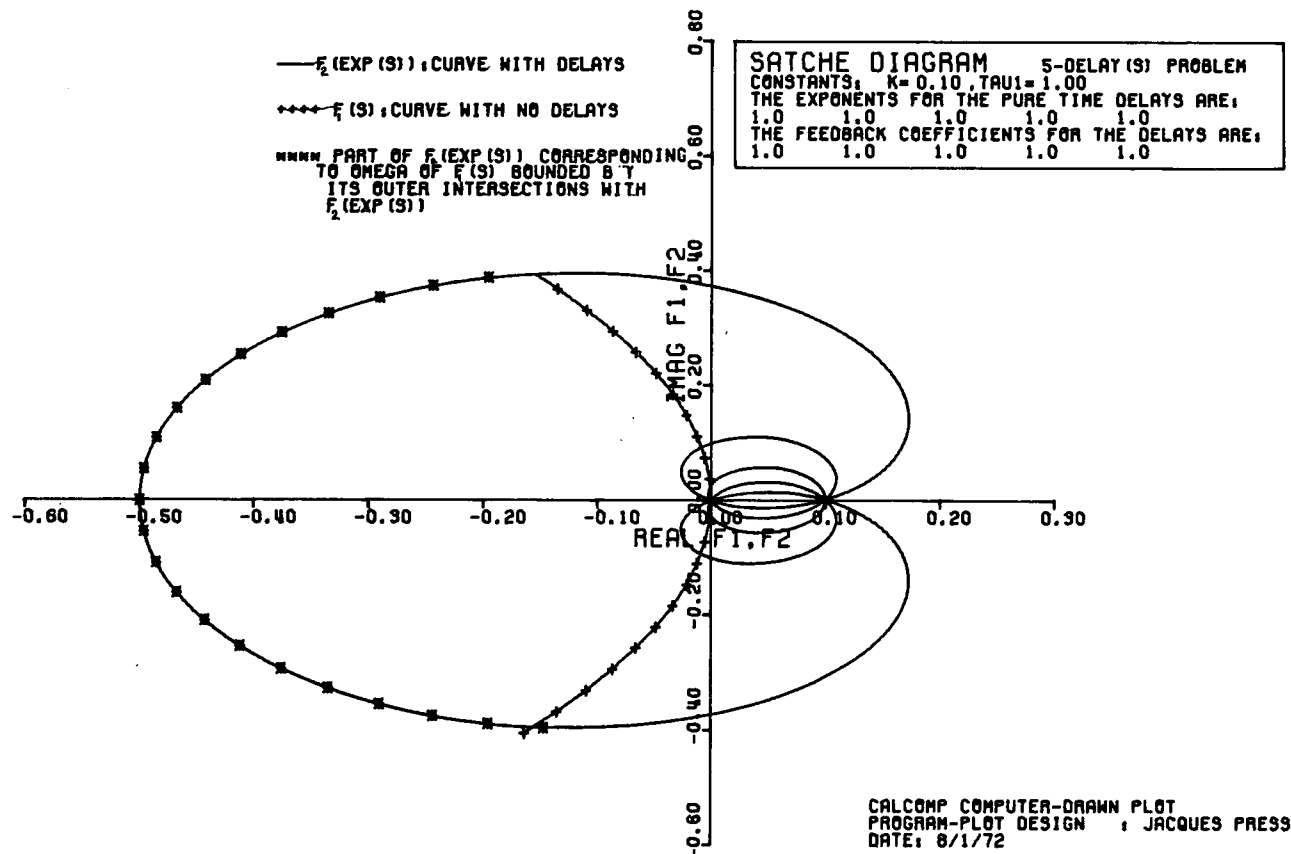


Figure 30. First Satche diagram for a case with five delays.

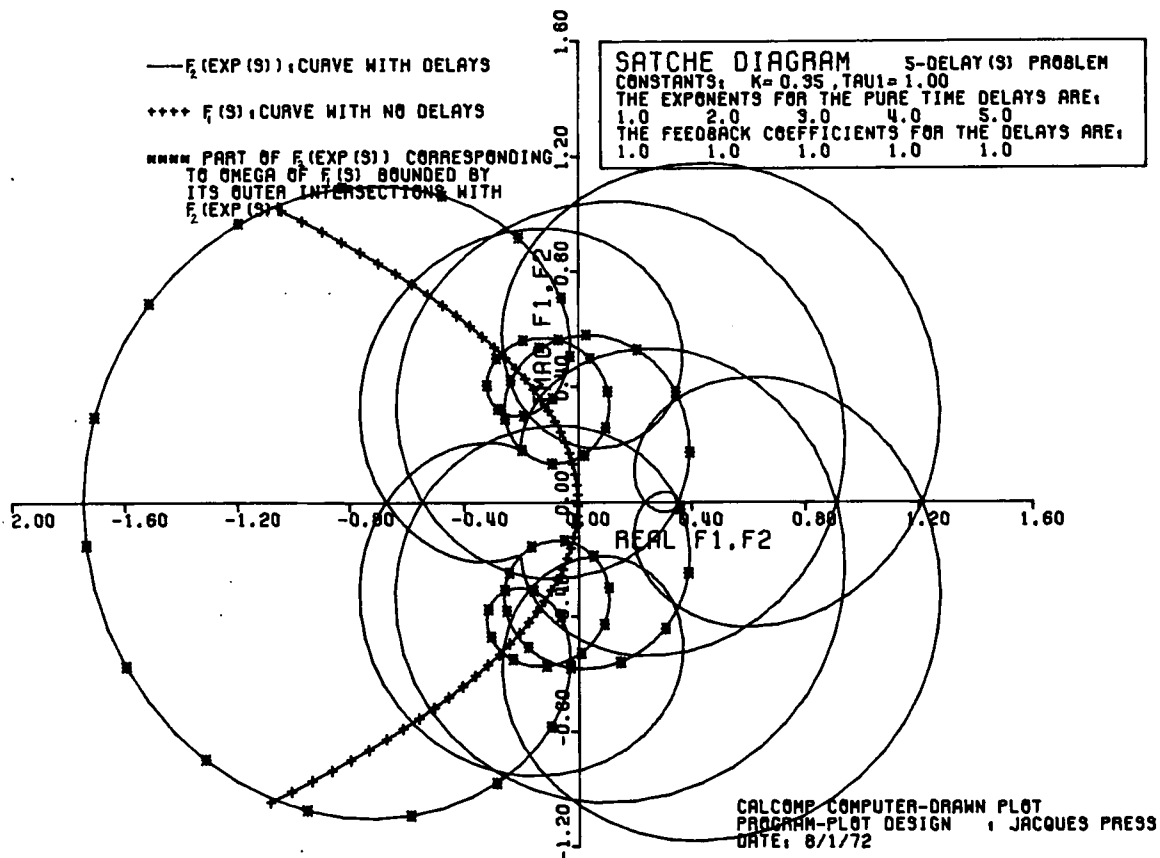


Figure 31. Second Satche diagram for a case with five delays.

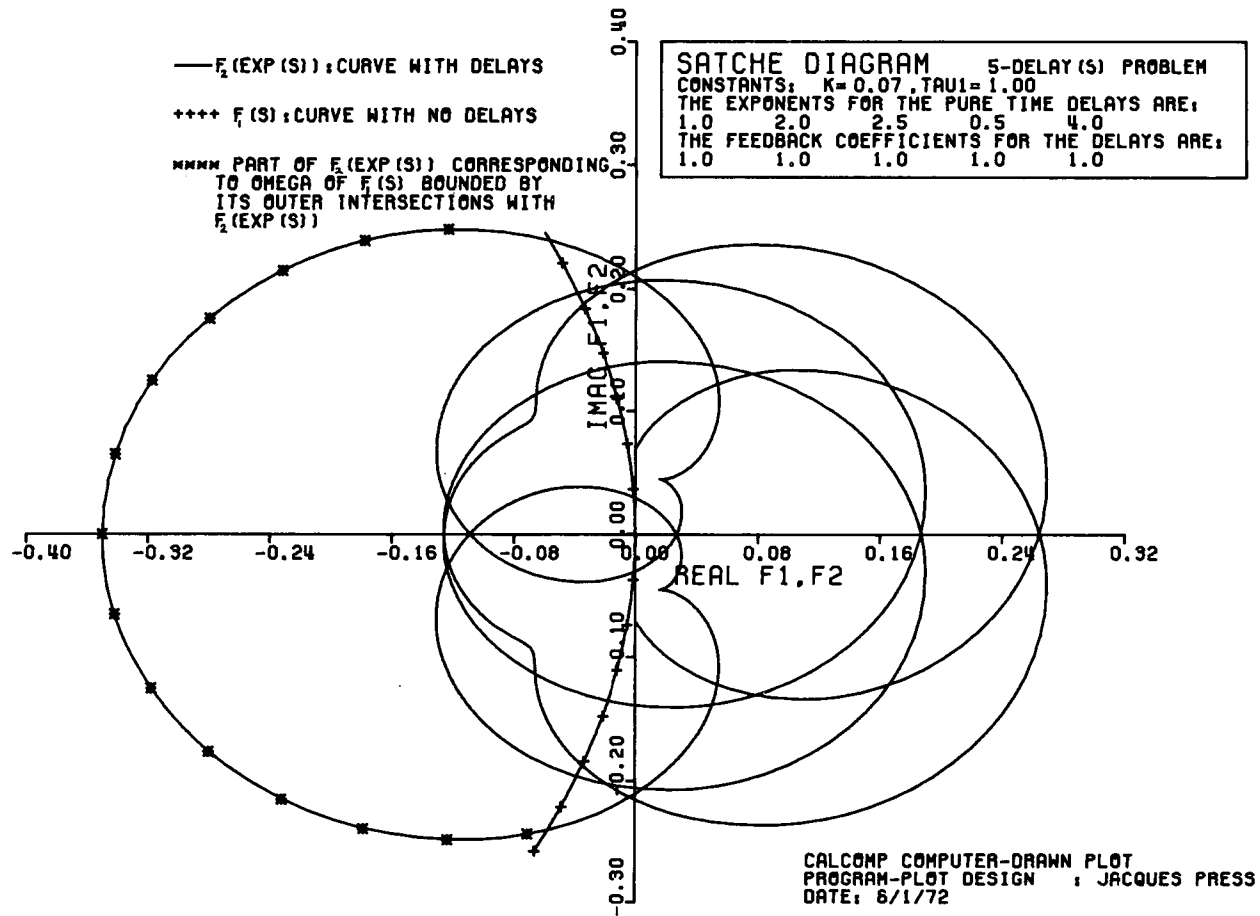


Figure 32. Third Satche diagram for a case with five delays.